THE

MATHEMATICAL GAZETTE

EDITED BY

T. A. A. BROADBENT, M.A.

ROYAL NAVAL COLLEGE, GREENWICH, LONDON, S.E. 10.

LONDON

G. BELL AND SONS, LTD., PORTUGAL STREET, KINGSWAY

VOL. XXXIII | MAY, 1949 | No. 304

HOW THEY LEARNT, 1600–1850.

Presidential Address to the Mathematical Association, 1949.

BY A. ROBSON.

ON some previous occasions when we have had schoolmaster presidents they have told us about the events of the early years of our Association.

Canon Wilson, who addressed us about twenty-five years ago, was an original member; he taught mathematics not far from here, at Rugby. Mr. Siddons told us of the beginning of mathematics in some well-known schools in the nineteenth century; he was himself a pupil of another of our original members, Rawdon Levett, who taught for thirty-four years at King Edward's School in this city. Levett was our first secretary, and when he retired the president attributed to him a full share of the credit for our initial successes. He was a schoolmaster of genius: one of the greatest teachers of his time. It is natural that we should think of him this year when we are meeting in Birmingham. Two years ago, Mr. Bushell, the son of another original member, told us about the last century of school mathematics.

What I want to do now is to convey to you some impression of the way in which elementary mathematics was learnt before the Mathematical Association or the A.I.G.T. existed, and chiefly how it was learnt in schools in England after the year 1600. It is necessary to include something of the work of universities, not only because 300 years ago little mathematics was learnt in schools, but also to notice the gradual trend of certain parts of mathematics down from the universities to the schools. I must avoid too great an overlap with the account we had of the work of the universities from Professor Chapman in 1946.

No doubt mistakes in teaching-method were made before the year 1600. Impositions, for instance, are said to have been used in Mesopotamia: but I have heard no evidence that their inventor was a mathematician. Much more recently than that, less than twenty-five centuries ago, Plato drew attention to a defect in primary education in Greece in the following words which I take from Heath's translation:

> " Freeborn boys should learn as much of these things as vast multitudes of boys in Egypt learn along with their letters. First there should be

> calculations devised as specially suitable for boys, which they should learn with amusement and pleasure ; for example, distributions of apples or garlands where the same numbers are divided among more or fewer boys, or distributions of the competitors in boxing or wrestling matches on the plan of drawing pairs with byes, or by taking them in consecutive order, or in any of the usual ways ; and again there should be games with bowls containing gold, bronze, and silver coins and the like mixed together, or the bowls may be distributed as undivided units. For by connecting with games the essential operations of practical arithmetic, you supply the boy with what will be useful to him later in the ordering of armies, marches, and campaigns, and in any case you make him more useful to himself and more wide-awake. Then again by calculating measurements of things which have length, breadth, and depth, questions on which the natural condition of men is one of ridiculous and disgraceful ignorance, they are enabled to emerge from that state."

Plato, one feels, might have had interesting suggestions to make about what should be done in secondary modern schools. What the effect of his remarks was at the time in Greece, I do not know. It can hardly have been comparable with the effect, at a higher level, of the appearance, about eighty years later, of Euclid's Thirteen Books of Elements, and that is a subject with which our Association has been much concerned.

Euclid's books almost at once superseded all previous books of Elements, including the works of those who consorted together in Plato's Academy and conducted their investigations in common : an early Teaching Committee. Thus we may regard Euclid's work as the definitive Greek pronouncement on the problems envisaged by Thales and the early Pythagoreans two or three centuries before. If Sir D'Arcy Thompson had lived to contribute to our 300th *Gazette*, we may conjecture that he would have developed the thesis that Euclid wrote an essay on the five regular solids for the use of initiates. We may like to think of it as a guide for students taking the third part of the Tripos in the University of Alexandria. One thing that Euclid did not do was to write a text-book for the use of preparatory schools. In the long run, the influence of his work was world-wide, but it is wrong to suppose that his books were the bane of the existence of schoolboys from 300 B.C. to A.D. 1903. It is true that some of the books were used in schools for a number of years, and for more years in England than elsewhere, but it is not for as many years as is sometimes supposed. There was first of all a period of eighteen centuries in which Euclid was available only in manuscript or in university lectures to a small number of students. The work was printed first, in Latin, at Venice in 1482 ; and the first English edition was published a hundred years later by Henry Billingsley, who had been a scholar of St. John's College, Cambridge. This edition consisted of 928 folio pages, besides the very fruitful preface of John Dee ; it is the edition on the title-page of which Euclid is wrongly described as Euclid of Megara. Dee was a Fellow of St. John's and afterwards of Trinity. He lectured on Euclid in English to a large audience in Paris in 1550. Afterwards he became an alchymist. It is not likely that copies of Billingsley's Euclid found their way into all the schools of England. If they had, perhaps the copy in King's School, Grantham, might have been noticed by one of the pupils ; but actually when Newton went to Cambridge in 1661 he knew no geometry. He afterwards used a copy of Barrow's abbreviated Latin translation of Euclid. Nor was King's School, Grantham, alone in not teaching mathematics. Samuel Pepys went to the grammar school at Huntingdon, and afterwards to St. Paul's ; he became a scholar of Magdalene College, and took his degree in 1654 ; but eight years after that, when he was a rising

young man in the Navy Office, he wrote as follows in his diary :

> "By and by comes Mr Cooper, mate of the *Royal Charles*, of whom I am to learn mathematiques and do begin with him to-day, he being a very able man and no great matter, I suppose, will content him. After an hour's being with him at arithmetique, my first attempt beihg to learn the multiplication table, then we parted till to-morrow."

What, then, was learnt in the schools at that time? Many of the grammar schools had been founded in the previous century. Naturally enough, grammar was learnt : it was Latin grammar, and some of the boys learnt to speak Latin as a living language. "No scholar who hath attained to be able to speak Latin, shall speak English, neither within school nor without it, when they are among the scholars of the same or a higher form." That is not a regulation of one of the great classical schools such as Eton or Westminster, but of a grammar school in a country town in the south of England. In a sense, in such schools, no mathematics was learnt, but there would usually be Reckoning, as well as Reading and 'Riting.

Taking one particular school founded in the middle of the sixteenth century, the original teaching staff consisted of a grammar master whose salary was £15 a year, a grammar usher at £10, and four other masters at £2 13s. 4d. each, namely a teacher to write, two masters to teach A B C to the petties, and a master of music to teach pricksong.

Newton once wrote : "A man may understand and teach arithmetic without any other skill in mathematics, as writing masters usually do." It was from these writing masters that arithmetic was often learnt, either within school or without it ; for these small-salaried masters sometimes eked out their living with private pupils. The "teacher to write" at the school of which I spoke, or one of his successors, was also teaching the children to cast accounts in the sixteenth century. A hundred years later one of his successors, being demanded whether he could teach common arithmetic, made a very slender answer, saying that if he were deficient in arithmetic, he would make it his business for the time to come to inform himself better in it ; and so he was told to come no more into the school after Christmas Day. On the other hand, another writing master, being of good life and conversation, was promoted to be an assistant to the mathematical master.

To see what kind of arithmetic was learnt, we may look at the *Ground of Arts*, "teaching the perfect work and practice of arithmetic, both in whole numbers and in fractions, after a more easie and exact form than in former times hath been set forth." This book was made by Mr. Robert Record, Doctor in Physick, and was dedicated to his most distinguished patient Edward VI, King of England, France, and Ireland. It was afterwards augmented by John Dee and others, and contains in some editions a section on sports and pastimes which might have earned the approval of Plato. Here is an example from it :

> "If a man doth cast three dice and you will know the points of each, cause him to double the points of one die, and to the double to add 5, and this sum to multiply by 5, and to the product add the points of one of the other dice, and behind the number towards the right hand to put the number which signifieth the points of the last die ; and then ask what number he keepeth, from which abate 250, and there will remain three figures which note unto you the points of every die."

There are, however, also in the book examples of the kind of rules to which the Mathematical Association took exception in its early days ; for example, concerning the extraction of the cubic root drawn out of the theorem which

Ramus made imitating that of Euclid concerning square root ; and there is also this :

> " If you pay for your board for three months 16s., how much shall you pay for eight months? "

" To know this you shall consider which of your numbers be of one denomination and set these, one over the other, so that the undermost be it that the question is of. As in my question, 3 and 8 be both of one denomination for they both be months, and because 8 is the number that the question is asked of, I set the one over the other and 8 undermost, thus $^{3}_{8}\mathrm{Z}$ with a crooked draught of lines. Then do I set the other number which is 16 against the 3 at the right side of the line, thus $^{3}_{8}\mathrm{Z}^{16}$, and to know my question I must multiply the lowermost on the left side by that on the right side, and the sum that amounteth I must divide by the highest on the left side, etc."

This is an example of the golden rule of three direct ; and when you consider that there is also a backer rule of three, a double rule of three direct and backer, and golden rules of three for fractions, you begin to appreciate why there used to be a jingle beginning with

" The rule of three, it puzzles me."

The *Ground of Arts* was not the first of the English printed arithmetics, but it was an early and successful one. About a hundred years after it came *Cocker's Arithmetic*, published in 1677, soon after Cocker's death, by John Hawkins, writing master near St. George's Church in Southwark, and commended to the world by him, and by Peter Perkins, John Collins, and others. Cocker was a practitioner in writing, engraving, and arithmetic, and he was also something of an optician, for he was consulted by Pepys about " some glass to help my eyes by candle-light ". The expression " according to Cocker " is still to be found in an English dictionary, though young people in a hurry prefer to say O.K.

Schools were sometimes careless about the teaching of arithmetic ; for example, John Wallis, who went to Felstead, learnt the subject, apparently by accident in the holidays, when he was fifteen. It took him a fortnight, but he had some help from a brother. In 1635 when Wallis was a third-year undergraduate at Emmanuel, he gave a gloomy picture of the state of mathematics at Cambridge. He had first to learn logic, ethics, physics and metaphysics, and—worse still—had to consult the opinions of the old schoolmen on those subjects. There had, in fact, been a series of losses at Cambridge. In the first place Thomas Gresham had disappointed his Cambridge friends by having his college built in London, and so there were mathematical professors in London before there were any in Oxford or Cambridge. Briggs went from St. John's College, of which he was a Fellow, to be the first Gresham geometry professor about 1596. A few years later, Oughtred, who was the best analyst in the University, also went down, to look after his country parish. Oughtred was an Etonian who became a Fellow of King's. He wrote a book on algebra and arithmetic called the *Clavis*, which was praised by Wallis and was used by Newton. He was also a pioneer in the use of abbreviations for sine, tangent, etc., and he gave the solutions of spherical triangles in all cases.

There was a college at Oxford—Merton—at which systematic instruction had been given in mathematics for two or three hundred years. When Henry Savile was Warden he delivered thirteen lectures, open to the University, on Greek geometry, reaching Euclid's eighth proposition and treating the work critically. In 1620 he founded the professorships which bear his name, and Briggs moved to Oxford to be the geometry professor.

Wallis, Seth Ward, and Charles Scarburgh, who were contemporaries at Cambridge, did, however, manage to learn some mathematics; they became Fellows of Queens', Sidney and Caius respectively. Ward is said to have brought mathematics into vogue again in the University. However, both he and Scarburgh were ejected from their fellowships for political reasons. Scarburgh went to Merton and studied medicine; also in 1649 Ward went to Oxford as Professor of Astronomy, and was afterwards President of Trinity College, Oxford, and Bishop of Salisbury. In the same year Wallis went to Oxford as Savilian Professor of Geometry, and Isaac Barrow became a Fellow of Trinity College, Cambridge. Barrow also was driven to spend some years on the continent, but he returned to Cambridge and became Professor of Greek. He became a Gresham professor in 1662, but shortly afterwards the Lucasian professorship was founded, and he occupied that Chair until Newton was ready to fill it. For the rest of the century mathematics flourished greatly at both Universities.

The time had come, too, for progress in some of the schools. In the 1670's it happened that there were several distinguished and influential men among the governors of Christ's Hospital in London. There was Jonas Moore, who had been tutor to the Duke of York and Surveyor-General of the Ordnance; he was the inventor of duodecimals and wrote a book on arithmetic in 1650, and he was one of the founders of the Royal Society.

There was also Samuel Pepys, who had become Secretary to the Admiralty and incidentally had found out how to do multiplication; for "Sir Jonas Moore told me the mighty use of Napier's bones, so that I am resolved to have a pair presently". Another governor was Charles Scarburgh, who had become Physician to the King; and there was the Surveyor of the King's Works, who was also Savilian Professor of Astronomy and the discoverer of the generators of a hyperboloid, Christopher Wren. At that time the Navy was badly in need of young people to be apprenticed to the captains of His Majesty's ships; and so it is not surprising that Charles II was persuaded by some of these governors to grant an additional charter and the promise of some financial help to their school. Accordingly they began from 1673 to select forty of the ablest and fittest youths in the school who had attained to a competent skill in grammar and common arithmetic as far as the rule of three, to be further instructed in arithmetic and navigation, to provide a mansion for the mathematical master to dwell in, and a convenient place where the children may be instructed in mathematics as aforesaid.

So that arithmetic was a recognised school subject before 1673. It was quickly found, however, that there was no suitable school text-book, and that no suitable course of further mathematics had been laid down.

Sir Jonas Moore set to work to write the book himself. It was called *A New System of the Mathematics* and was in nine sections, of which Moore had written four when he died. As this book was written nearly 200 years before the A.I.G.T. was founded, and as it embodies some of the principles for which our Association worked in its early days, it may be of interest to give a short description of its contents.

1. *Arithmetic*. This begins with a summary of the elements, "so that you children may have wherewith to mind you if it happens that you forget any of the rules". In the later part the explanations are concise but clear. Record's rule of three problem would have been set out without much fuss in the form 3.8 :: 16.q as in Oughtred, or as indeed we might set it out to-day. Decimals are explained as a more easie sort of fractions. Logarithms are numbers which differ arithmetically as the numbers answering to them differ geometrically. The word "index" is used instead of the word "characteristic" used by Mr. Briggs: the index indicates the number of places that

the leading figure is away from the units figure. There are some examples of interest and annuities taken from Oughtred.

2. *Practical Geometry.* "This is the second step by which you are to ascend into the admirable knowledge of mathematics. I defer teaching you the theory which you will learn in a later part. I have not gone the usual way, but I hope a better: more intelligible and fitter for life".

Explanations, less formal than Euclid's definitions, are given, beginning with the circle and straight-lined figures and the ellipsis derived from the string-and-foci construction, which is shown in a diagram. Here followeth the resolution of various geometrical problems fitting to be done for several uses, and also for the exercising of the hand in the management of the rule and compasses. Solid figures are soon introduced; e.g. the spheroid, which is made by turning the ellipsis about his major diameter: the ends cut off, you have a hogshead. Besides prisms and pyramids, there are the five figures called Plato's: the figures in the diagrams, being half cut-through with a penknife, will fold up and make these regular solids.

There are then mensuration examples. In these first two sections, there are several tables, e.g. for the conversion of decimals of various quantities, and there are some logarithms; but for the later part of the book a table of Mr. Briggs must have been available. The last table in Section 2 has a utilitarian sound, being no less than a table for converting solid inches into gallons of beer.

3. *Trigonometry*, plane and spherical, chiefly solution of triangles. It cannot be said that all the recommendations of the M.A. *Report on Trigonometry* are followed. There has been a delay in the publication of the *Report*: if it had appeared 300 years sooner, no doubt Sir Jonas would have hesitated to define two of the trigonometrical functions almost simultaneously. Those introduced first are the right-sine and the versed-sine, otherwise called the sine and the sagitta. They are defined for angles up to 180° with no sign-conventions.

The word "analogy" is used with its Greek meaning of proportion where to-day we would use the word "formula". It is still in the twentieth century occasionally used in spherical trigonometry, in rather a puzzling way, because the formulae to which it is applied are not usually expressed as proportions.

In a plane triangle, of the four quantities, two sides and the angles opposite to them, any three being given, the fourth may be found. The analogy to be used is: as one side is to the other, so is the sine of the angle opposite the first to the sine of the angle opposite the second. This problem is to be done in three ways, by scale-drawing, by sines, and by logarithms. Proofs of the harder analogies are deferred.

4. *Cosmography*, i.e. the description of the heavens, the stars, and the earth, is divided into two parts, astronomy and geography.

This subject, Moore says, is justly accepted as one of the liberal sciences. I suppose it is still so accepted to-day; but is it not very often unjustly neglected?

There is a description of the constellations, intended to be used with a star-globe. For example: Cassiopeia is the mother of Andromeda; she sits in her chair and has in her breast a star of the third magnitude called Schedar. The Harp, of 15 stars, has one of the first magnitude called Lucida Lyrae. The northern constellations and the constellations of the zodiac are first described. There are also twelve constellations round the south pole, which I expect shall shortly be rectified and adjusted as to their positions by the able and industrious Edmund Halley, now residing at St. Helen's Island for the purpose. Halley was twenty-four at the time; he had been at St. Paul's School and Queen's College, Oxford; and he afterwards succeeded Wallis as

Professor and Flamsteed as Astronomer-Royal. On account of the death of Jonas Moore, it fell to Halley to revise and pass through the press the portion on geography.

5. *Navigation*. This section occurs naturally on account of the special aims of the school for which the book was chiefly designed. It was written by Peter Perkins, the master at the school. Evidently, with him, navigation was by no means an indoor subject. On one occasion he got into trouble for neglecting to inform the under-nurse of the evenings on which he required the children to sit up to take their observations of the rising and down-going of the moon and stars. One of the problems is this: "Of the fore-staff, a vane or cross being lost, to find what length it was." It is easy to imagine that with twenty boys on the roof, one of the vanes would be lost: it might have been needed to attract the attention of a passer-by in Little Britain, near to the place where the G.P.O. now stands. Perhaps they had to make a new vane on the next day.

6. *Doctrine of the Sphere*, or, as we would say, of the Universe.

This was written by John Flamsteed, Astronomer-Royal. Among other things it describes what it calls the ancient Pythagorean theory lately revived by Copernicus, and compares it favourably with the new Aristotelian theory that was adopted by Hipparchus and Ptolemy and is now generally abandoned. There are some remarks about finding longitude at sea by the method of lunar distances, which to my regret at the time was not quite obsolete when I first learnt Navigation in the mathematical school at Christ's Hospital. Another passage (about eclipses) may have some historic interest: "I first light on this method," says Flamsteed, "in 1676. I transmitted it to my friend Jonas Moore, who communicated it to the Royal Society at one of their meetings. Christopher Wren being there present and having only viewed the figure told him that himself had known the method 16 years agone, and to assure him of it, sent him a like projection neatly drawn on pasteboard and fitted with several ingenious contrivances of numbers and scales for the construction of solar eclipses in our latitude. This Sir Jonas brought to me then labouring under some distempers at Greenwich, whereby I was satisfied that the invention was due to Sir Christopher Wren, whom of all men I believe to be the first to find the beginning, middle, digits then darkened, inclination of the cusps at any phasis, and the end, of a solar eclipse, without the calculation of parallaxes."

Flamsteed makes some apology for the difficulties of the eclipse section, but says that where there are such able teachers as Peter Perkins to explicate them, it may be an advantage if the work is not too easy.

7. *Astronomical Tables* were also provided by Flamsteed. These included equations of the apparent time; recess of the equinoxes from ♈; the earth's and the moon's mean motion for each day of the year; and the equations of the earth's orbit. This was eighty-five years before the publication of Maskelyne's first *Nautical Almanac*.

The last two sections, 8. *The Arithmetic of Species or Algebra*, and 9. *The Chief Propositions of Euclid*, had originally been intended to be written by Perkins: but Perkins died within a year of Moore and it appears to me that Sections 8, 9 as published were makeshifts. Section 9, instead of being a selection of the important propositions, was an English version of Barrow's *Euclid* limited to Books I to VI, XI, XII, and such versions were available at the time. In another *System of Mathematics* written by the master at the school in 1723, the original plan was carried out: about twenty of Euclid's propositions, including some of the constructions, are given in full, and others are treated as corollaries with short indications of method; hypothetical

constructions are used and algebraical explanations are given for most of the propositions of Books II and V. None of XI, XII are included.

The algebra section is quite short. In it, Oughtred's notation $a \cdot b :: c \cdot d$ is used and is transformed into $ad = bc$ by a forward reference to a proposition of Euclid's Book VI. There is a separate supplement about surds; otherwise the algebra ends with questions producing simple equations, e.g.

> "A certain man, finding diverse poor people at his door, gave each of them 3d. and had 6d. remaining in his pocket. But if he *would* have given each of them 4d., he must have *wanted* 2d. How many poor were there?"

In some early syllabuses, Sections 1 to 7 were followed quite closely, but there is no reference to Sections 8, 9, and it may well be that these were not used. A Latin algebra was presented by a benefactor, and this was afterwards translated into English by the under grammar master for ten guineas.

The question of Latin was becoming an issue in the schools as early as 1681; for when Moore's book was first published it was welcomed by the governors, some of whom, however, expressed the hope that as soon as possible it would be translated into Latin. A little later, when Newton was a governor and was asked for his opinion on a syllabus, he replied that it could not be covered thoroughly in two years, and he could not but regret therefore that the children should spend five years learning Latin. Apparently a spirit of compromise prevailed, for while Moore's book was not put into Latin, it was ordered that the children should have copies of Beza's Latin translation of the Testament and Latin Books of Common Prayer to make use of at church for the preservation of their Latin. More effective perhaps was the regulation that they should attend in the grammar school on Wednesday afternoons. It was a long time before the mathematical school was merged with the other schools, but in 1715 the senior boys from the grammar school began to go to the mathematical school for two hours a day, namely, from 7 to 9 a.m., that they might be better prepared for the university.

Another remark of Newton's of general interest was that there was a great lack of good mechanics teaching, to which he attributed the advantage the French have over other nations in their engineers. Some attempt was made to remedy this with the help of Humfrey Ditton.

The problem of finding mathematical teachers is also not only a twentieth-century problem. It was as urgent in 1686, when it took twenty-eight years to find a worthy successor of Peter Perkins. In the time of his immediate successor, the children were found to be very deficient in answering to what Dr. Wood said that he had taught them. Dr. Wood explained that he had had an ague which had prevented him from attending the children; he had, however, employed another man to look after them. On enquiry, however, it was found that the other man was one who, by the manners of his conversation, indecencies of habit, looseness of manners, and public exposing of his intemperance to the children, had forfeited all sort of awe from them to his discipline.

When you consider what was required of a mathematical master in those days, you will understand why it was hard to find a "sufficient master", and you may wonder whether you would have qualified for the post.

He was to be a sober, discreet and diligent person of good life, government and conversation; a good scholar well understanding the Greek and Latin languages, so that the boys be furthered in the Latin tongue and the master able to answer strangers in that language; he should write a good scrivenor hand, that he may be to the children as good as a writing master, and he must be a very good mathematician well knowing in the theory and practice

of all its parts, and so ready that no stranger from abroad or practitioner at home may be able to baffle him.

I have dwelt at length on that particular school, which had the special advantage of its naval apprentice scheme ; but it is not easy to find details about what happened at other schools 250 years ago. Some of you may be interested to investigate conditions at that time in the schools where you have learnt or taught. There were certainly other schools where progress was made. At King Edward's School, Birmingham, there is a record of the appointment of a mathematical master, Anthony Thacker, in 1743.

Sir Joseph Williamson's school at Rochester was founded early in the eighteenth century. It had a lower master who taught the boys arithmetic until they could perform readily the several operations thereof, whether in whole numbers, vulgar, or decimal fractions, together with the common and obvious uses of the same and the extraction of the square and cube roots. When the boys were ready to learn geometry and fit for further instruction in mathematics, they went to the upper master, who taught them the Latin and Greek authors, rhetoric, and the rudiments of mathematics such as algebra, geometry, trigonometry and the use of the globes, and gave them such insight into the ancient and modern geography as was necessary for the understanding of the authors they read. The first upper master was John Colson, afterwards Lucasian Professor.

These school courses include the subjects which were studied at the universities in 1650, but the school treatment would be more practical if Jonas Moore's views were accepted. Meanwhile there was rapid progress at the universities, and it is interesting to compare Wallis' account of 1635 with Newton's ideas of what ought to be done at his college at a time when Wallis was still living in Oxford. There is no reason to think, however, that Newton's scheme was ever put into effect in all its grim efficiency : Each undergraduate to be taught by a tutor and four lecturers. The tutor to read logic, ethics, the globes, geography, and chronology in order that history may be understood; two lecturers to set tasks in Latin and Greek once a day in the first year and once a week to the rest, and to appoint the reading of the best historians ; and to set bigger tasks in the vacations proportionally to the extra spare time of the students. The philosophy lecturer to read first introductory matter : time, space, motion, force, gravity, mechanical powers, hydrostatics, solid and fluid projectiles, and circular motions ; and then to read natural philosophy, starting with the general system of the world and proceeding to the particular constitution of the earth and things therein : elements, minerals, vegetables, animals, and ending with anatomy if the lecturer be skilled therein ; and this lecturer is to examine in ethics and logic. The mathematics lecturer is to read first some easy and practical things, and then Euclid, spherics, projections of the sphere, maps, trigonometry, astronomy, music, optics, and algebra ; and to examine in chronology and geography and to instruct in them if the tutor be deficient. The tutor to take care that his pupils read over all last year's lessons in the long vacation ; at the end of it they shall be examined, and those not fit to go on shall be turned down to the lectures of the year below. Several sciences which depend not on one another are learnt in less time together than successively, the mind being diverted and recreated and put more upon the stretch.

Rather later than this, the subjects actually studied were stated as

1st year : Arithmetic, logic, Euclid, geography, philosophy ;
2nd year : Astronomy, conics, physics, philosophy ;
and then : Newton's Optics and more astronomy and philosophy.

But the eighteenth century was not a time in which the English universities

greatly flourished. There was much to be done in assimilating the discoveries of Newton and others; and attention began to be given to the calculus, then, in England, called the theory of fluxions. You will remember the maxim given by Chrystal in the preface to his *Algebra*: "Go on, but often return to strengthen your faith." In the eighteenth century the continental mathematicians went on, but the English paid more attention to the second part of the maxim. Also, by adhering too closely to Newton's notation and methods, they lost touch with the continent. In 1734 Bishop Berkeley launched an attack on the foundations of fluxions in his pamphlet called the *Analyst*, a discourse addressed to an infidel mathematician, wherein it is examined whether the object, principles and inferences of the modern analysis are more distinctly conceived or more evidently deduced than religious mysteries and points of faith. One of the first replies to this was called *Geometry no friend to infidelity*, addressed to the author of the *Analyst*, wherein it is examined how far the conduct of such divines as intermix the interest of religion with their own private disputes and passions, and allow neither learning nor reason to those they differ from, is of honour or service to Christianity or agreeable to the example of our Saviour and his apostles. This was written by James Jurin, who went to Cambridge in 1702 from Christ's Hospital and was a pupil of Newton's. He became a Fellow of Trinity and studied medicine at Leyden. He became an F.R.S. and President of the Royal College of Physicians. Other replies were written by a self-taught mathematician, Benjamin Robins, and by Walton, the Professor at Dublin. The most effective answers were some better presentations of the subject. John Colson published a translation, with a commentary, of Newton's *Method of Fluxions*, and other accounts were written by John Müller, head of the R.M.A. at Woolwich, and by James Hodgson, F.R.S., who was mathematical master at Christ's Hospital for forty-six years from 1708.

More important was the *Treatise of Fluxions* of Colin MacLaurin, the professor at Edinburgh, which was being written at the time of Berkeley's attack, but which was delayed until 1742 for the completion of Part II. MacLaurin's plan was to represent fluxions by finite quantities, but because the use of infinitesimals is valued for its conciseness, he thought it requisite to account explicitly for the truth and perfect accuracy of the conclusions that are drawn from it, avoiding certain expressions which, though convenient, might occasion disputes. Nevertheless the controversy continued, and so did the publication of books on fluxions. One of these by Vince was part of James Wood's four volumes of the *Principles of Mathematics and Natural Philosophy* published in 1795 and much used at Cambridge.

Before the eighteenth century, university tests for degrees were entirely oral. Bentley while Master of Trinity reformed his college fellowship examinations; that was in Jurin's time. During the century the Tripos gradually ceased to be oral, though it was not entirely written until about 1835. A small but useful change of procedure was made in 1763 when Paley was Senior Wrangler, when there was a strong-minded moderator called Richard Watson. Watson's own career illustrates what it was possible to do in those days in breadth of subjects. He went to school at Heversham in Westmorland, where he says in his memoirs that he was not given a sound classical education, not being taught to write Greek or Latin verses; he says nothing about what mathematics he learnt at school, but he was second wrangler in 1759 when, he says, the leading moderator made a person of his own college first in direct opposition to the general sense of the other examiners. But his old master, Dr. Smith, told him not to be discouraged, for that when the Johnians had the disposal of the honours, the second wrangler was always looked upon as the first. As an undergraduate Watson studied mathematics

in the morning and classics in the afternoon, and he was ashamed to say that he sometimes wasted an evening. He became a fellow and tutor of Trinity, and was made professor of chemistry at a time when he had not read a syllable on the subject, nor seen an experiment performed ; but he was tired of mathematics and natural philosophy. He lectured and wrote on chemistry and became an F.R.S. When the divinity professorship became vacant, Watson was puzzled for a moment ; for although he knew as much divinity as could be expected of one whose time had been fully occupied with other subjects, he was not even a bachelor of divinity, and the regulations required that the professor should be a doctor. There was only a week to spare. However, by some hard travelling and a little adroitness, he managed to obtain the King's mandate for the D.D. degree in six days. He was professor of divinity for forty-five years, during the last thirty-four of which he was also Bishop of Llandaff, and for the last twenty-five he practised arboriculture on the shores of Windermere.

The development of the Tripos can be gauged by looking at the problem papers for 1785, 1786, which are reprinted by Rouse Ball in one of his books. A few specimen questions will show what was expected of the best men at that time :

1. Reduce the biquadratic equation $x^4+qx^2+rx+s=0$ to a cubic equation.
2. Prove that the asymptotes of a hyperbola are external to the curve.
3. Find the fluent of $\dot{x}\sqrt{(a^2-x^2)}$.
4. Suppose a body thrown from an eminence upon the earth, what must be the velocity of projection to make it become a secondary planet to the earth?
5. What is the relation between the 3rd and 7th sections of Newton, and how are the principles of the 3rd applied to the 7th?

On the other hand, the requirements for a second-class degree were less than those for the Previous Examination a hundred years later.

Among the senior wranglers of the 1790's was Robert Woodhouse, who took an important place in the next stage of the reform of university mathematics. There was also George Butler, who became headmaster of Harrow in the year of the battle of Trafalgar, and who taught Euclid to his sixth form. He may have used Simson's *Euclid* ; in the edition of this which appeared in 1781, it appears from the preface that trigonometry was taught after the elements of Euclid.

In the last part of the eighteenth century and up to about 1820, many of the grammar schools were in a state of decay. Richard Watson's school, the small grammar school at Heversham, seems to have been an exception. In the seventy years from 1750, twenty of its scholars became fellows of their colleges, ten at each university. These included a senior wrangler, three second wranglers, and a first class Oxford mathematician who became an examiner in "Greats". The best known of these was William Whewell, who became Master of Trinity. He *was* taught to write verses at school, and he afterwards said that he was given a good foundation of arithmetic, mensuration and practical geometry. At the age of 17½, on the death of his headmaster, he was put in charge of the school until he went to Cambridge. In that period he learnt some mechanics, conics and fluxions from a blind mathematician, Charles Gough, who lived at Kendal. But, according to Todhunter, writing cautiously in 1876, one of Gough's solutions was not in accordance with the principles used by writers on mechanics for problems about the collision of hard bodies.

Dissatisfaction with the curriculum in some of the grammar schools led to the foundation of new schools and ultimately forced reforms in the old ones. It was not always easy to make the changes. To take one example, John Scott was at the grammar school at Newcastle about 1765, and was taught by a master painful to instruct the youths in Latin and Greek. Scott, who was afterwards Lord Eldon, gave some decisions (not wholly approved by other lawyers) to the effect that only Latin and Greek could legally be taught in certain grammar schools. It was, in fact, only in 1820 that instruction began to be given in his Newcastle school in English, History, Geography, the use of the Globes and Mathematics.

When Whewell first went to Cambridge, it is said that he walked from Lancaster. That would be in 1812. It was while he was an undergraduate that the writings and lectures of Woodhouse began to take effect. Woodhouse wrote a book called the *Principles of Analytical Calculation* in which he explained the differential notation used on the continent, though he criticised some of the fundamentals of calculus on that basis. The initiative was taken by some undergraduates, including John Herschel, George Peacock and Charles Babbage, and the Analytical Society which they formed published a translation of a French calculus text-book. The changes were pushed by George Peacock when he became a moderator, and he was then supported by Whewell in his mechanics book and by Airy. By 1830, when a new book on analytical geometry was written, the new methods had secured a place alongside the old. Woodhouse also wrote an astronomy, including an account of the physical theories of Laplace and others, thus again helping to restore contact between England and the continent. He wrote a trigonometry, one passage of which must have been read by many students up to recent times. After showing how to use subsidiary angles and trigonometrical tables to solve quadratic and cubic equations, he continues: "If in mathematical researches equations like these presented themselves to be solved, their solution would be conveniently effected by these methods; but the truth is that in the applications of mathematics to physics the solution of equations is an operation that very rarely is requisite, and consequently the preceding application of trigonometrical formulae is to be considered as a matter of curiosity rather than of utility." This was quoted by Todhunter in his text-book of 1874, and was retained in the revised edition of it prepared by R. W. Hogg after Todhunter's death. No doubt many of you have used it. It is strange how long we sometimes take to learn: in his *Introduction to Mathematics*, written a hundred years after Woodhouse's trigonometry, Whitehead is at pains to point out that too large a place is generally given in algebra to solution of equations.

At Cambridge, the Classical Tripos was instituted in 1824, but until 1850 candidates had first to take the ordinary tripos and obtain at least a "third" in it. In 1800 this would probably have been a very slight grievance; but by 1850 the standard of the mathematics had risen considerably. A good account of this period is given by the American, Charles Astor Bristed, who was at Cambridge for five years from 1840 after taking a degree at Yale. This shows how, in the various college and university examinations, mathematicians and classics were tossed about from one subject to the other, not knowing just how little they could afford to do of the subject they liked least. There was no university entrance examination; for although the Previous Examination started at the same time as the classical tripos, it was a second-year examination. Bristed, being a classic, was rather alarmed at having to spend some hours in a college examination solving quadratic equations, writing out the *Pons Asinorum* by ἀνάμνησις, and answering questions about Paley's theology; but he had done these subjects in America. It was not so

easy to qualify in the mathematical tripos. He came 112th, missing the wooden-spoon by two places. To do this he had to study Books I to IV, VI, XI of Euclid; three sections of Newton's *Principia*; algebra to theory of equations; conic sections, and statics, dynamics, hydrostatics, optics, so far as these could be done without calculus. Some of these subjects were done quite differently in America; for instance, in England sines, cosines, etc., were ratios, not lines, and the conics were treated analytically in a book by Parr Hamilton.

On a visit to Oxford, Bristed estimated the difference between the universities to be that whereas at Cambridge men did classics and mathematics, at Oxford they did classics and logic; also, a relic of a cholera epidemic, in Oxford confectioners were not allowed to deliver ice-cream to the students' rooms. On the other hand, Frederick Temple, afterwards Archbishop, who was at Oxford at the time, said: "The only thing I have to complain of at Oxford is the low state of science. They have not a single book on logic worth reading, and metaphysics and moral philosophy are at a very low ebb. Mathematics are still lower." Class-lists for the degree examinations began to be published at Oxford in 1807, and classics began to be separated from mathematics from 1831.

When one reads that Dr. Arnold caused his classical masters also to teach mathematics at Rugby, one has to remember that they might have been perfectly competent to do it. The "grave young master" in *Tom Brown's Schooldays* was eighth in the classical tripos, but he was also a senior optime, and in fact his degree was much the same as that of the first master appointed at Winchester specifically to teach mathematics, and who was universally agreed to have been an excellent teacher of it. But when, after 1850, the supply of men with double qualifications, even from Cambridge, began to dwindle, and also the standard of school mathematics rose, specialist teachers became a necessity; and so did our Association.

In our early days, the reform of the teaching of geometry was our preoccupation; and it is still an important part of our work. It is admitted now that we start with some easy and practical things; but some of us think that progress since 1903 has not been sufficiently rapid.

Another subject that concerns us is calculus. Roughly speaking, calculus was invented in the seventeenth century, thought about in the eighteenth, taught at the universities in the nineteenth, and only gained a firm footing in the schools in the twentieth. How has it come about that it is now taught in fifth forms? Possibly this is partly due to the increase of rigour in the universities: for if we are going to put across an elementary course in the schools, we should start by knowing very well where the ice is thin.

An improved liaison between school and university teachers helps our reforms more than we realise. Let me end with a plea for better contacts between teachers of all sorts: for example, it may be that teachers in grammar, modern and technical schools have much to learn from one another in the near future. We must remember the aim of our Association to bring within its purview all branches of elementary mathematics. A. R.

ON THE DETERMINANTS OF PAN-MAGIC SQUARES OF EVEN ORDER.

By N. Chater and W. J. Chater.

In Mathematical Note 1911, this journal, Vol. XXX, No. 290, A. H. MacColl forms the determinant Δ of a pan-magic square of order 4 and states that $\Delta = 0$. He also remarks that "the property appears to be peculiar to pan-magic squares of order 4". Below we give our demonstration of the above theorem and extend it to the determinants of all pan-magic squares of even order. We shall show that the determinants of all pan-magic squares of even order have zero as their value. In the proof that follows below we make use of a property of pan-magic squares of even order.

As is well known, the pan-magic square of order 4 has the sum of the quantities in alternate diagonal cells equal to $\frac{1}{2}S$, where S is the square's constant. That is, starting at a cell containing a quantity a, we reach a quantity A in an alternate cell set diagonally to the first such that: $a + A = \frac{1}{2}S$.

This is true wherever in the square we start our a. Between the cell containing a and that containing A there is one intervening cell. Quantities such as a and A whose sum equals $\frac{1}{2}S$ are spoken of as "complementary quantities". This position of any quantity a of the square relative to its complement A allows us to fill in the whole square thus:

a	b	c	d
e	f	g	h
C	D	A	B
G	H	E	F

where $a + A = b + B =$, etc. $= \frac{1}{2}S$.

It is this well-known property that we now generalise in order to use it afterwards in our demonstration.

Extension to pan-magic squares of order 6.

Here the corresponding property enables us to fill the cells thus:

a	b	c	d	e	f
g	h	i	j	k	l
m	n	o	p	q	r
D	E	F	A	B	C
J	K	L	G	H	I
P	Q	R	M	N	O

The quantity a has its complement A in the 3rd diagonal cell from it or, stated otherwise, two intervening cells separate a and its complement A.

In the square of order 6, $a + A = b + B = \dots$, etc. $= \frac{1}{3}S$.

Square of order 8.

In a pan-magic square of order 8 the complementary quantities lie along diagonals, full or broken in cells separated from each other by three cells, or A lies in the 4th diagonal cell away from a.

Here $$a + A = b + B = \dots, \text{ etc.} = \tfrac{1}{4}S.$$

We may reach A from a by following sometimes a full diagonal and sometimes by following a broken one, but whichever diagonal path we take in a square of order 8 there are always three cells separating a from its complement A, or A always lies in the 4th diagonal cell from a.

The table below sets out this property of pan-magic squares of even order.

Order of square	4	6	8	$2N$
Number of cells along a diagonal, full or broken, separating a quantity a from its complement A.	$\frac{4}{2} - 1 = 1$	$\frac{6}{2} - 1 = 2$	$\frac{8}{2} - 1 = 3$	$\frac{2N}{2} - 1 = N - 1$
Sum of any quantity such as a and its complement A.	$\frac{1}{2}S$	$\frac{1}{3}S$	$\frac{1}{4}S$	$\frac{1}{N}S$

This property divides all even ordered pan-magic squares into four parts by lines through the middle points of its sides—quadrants. Thus on a square of order 6 the top left-hand quadrant contains the quantities :

a	b	c
g	h	i
m	n	o

while the lower right-hand quadrant contains the corresponding complementary quantities :

A	B	C
G	H	I
M	N	O

with two other similarly filled quadrants. And so on for higher order squares of even side.

In all such squares the sum of the quantities in each of the four quadrants is the same. It equals $\frac{6S}{4}$ for a square of order 6 and $\frac{8S}{4}$ for a square of order 8.

We have demonstrated the general property of the above table in a paper shortly to be submitted for publication.

We now make use of the foregoing in order to demonstrate the property announced in Mr. MacColl's note and show that it may be extended to the determinants of all pan-magic squares of even order.

We take the 4×4 case first.

Let
$$\Delta = \overset{(1)}{\begin{vmatrix} a & b & c & d \\ e & f & g & h \\ C & D & A & B \\ G & H & E & F \end{vmatrix}}$$
be the determinant of a pan-magic square of order 4.

Then
$$\Delta = \overset{(2)}{\begin{vmatrix} C & D & A & B \\ G & H & E & F \\ a & b & c & d \\ e & f & g & h \end{vmatrix}} = \overset{(3)}{\begin{vmatrix} A & B & C & D \\ E & F & G & H \\ c & d & a & b \\ g & h & e & f \end{vmatrix}}$$
where (2) is obtained from (1) by interchanging rows 1 and 3 and 2 and 4, and (3) is obtained from (2) by interchanging columns 1 and 3 and 2 and 4.

Thus Δ has the two equal forms (1) and (3). The proof takes these two determinants (1) and (3) and shows that $(1) = X$, say, and that $(3) = -X$, whence $\Delta = \pm X$ and so must vanish.

Treatment of Determinant (1).
$$\Delta = \overset{(4)}{\begin{vmatrix} S & b & c & d \\ S & f & g & h \\ S & D & A & B \\ S & H & E & F \end{vmatrix}} = \overset{(5)}{\begin{vmatrix} 4S & S & S & S \\ S & f & g & h \\ S & D & A & B \\ S & H & E & F \end{vmatrix}}$$
where (4) is obtained from (1) by adding columns 2, 3, 4 to column (1) and where (5) is obtained from (4) by adding rows 2, 3, 4 to row 1.

Treatment of Determinant (3).
Similar treatment of 3 yields:
$$\Delta = \overset{(6)}{\begin{vmatrix} 4S & S & S & S \\ S & F & G & H \\ S & d & a & b \\ S & h & e & f \end{vmatrix}}$$

We shall now show that when, in a determinant bordered by S's such as (5) or (6) above, we change quantities of a row such as F, G, H into their complements f, g, h, only the sign is altered, not the value.

Take (5). This equals:
$$\begin{vmatrix} 4S & S & S & S \\ S & f & g & h \\ 2S-S & \tfrac{1}{2}S-d & \tfrac{1}{2}S-a & \tfrac{1}{2}S-b \\ S & H & E & F \end{vmatrix}$$
$$= \begin{vmatrix} 4S & S & S & S \\ S & f & g & h \\ 2S & \tfrac{1}{2}S & \tfrac{1}{2}S & \tfrac{1}{2}S \\ S & H & E & F \end{vmatrix} - \begin{vmatrix} 4S & S & S & S \\ S & f & g & h \\ S & d & a & b \\ S & H & E & F \end{vmatrix}$$
$$= 0 - \begin{vmatrix} 4S & S & S & S \\ S & f & g & h \\ S & d & a & b \\ S & H & E & F \end{vmatrix}$$

The quantities D, A, B are changed to d, a, b with change of sign only.

Similar treatment converts the last determinant into :

$$\overset{(7)}{+\begin{vmatrix} 4S & S & S & S \\ S & f & g & h \\ S & d & a & b \\ S & h & e & f \end{vmatrix}}$$

where H, E, F are changed to h, e, f with change of sign only.

Taking determinant (3) and treating it as above, we have :

$$\Delta = \begin{vmatrix} 4S & S & S & S \\ S & F & G & H \\ S & d & a & b \\ S & h & e & f \end{vmatrix} \text{ and then : } \overset{(7)}{-\begin{vmatrix} 4S & S & S & S \\ S & f & g & h \\ S & d & a & b \\ S & h & e & f \end{vmatrix}}$$

Thus $\Delta = \pm$determinant (7), whence $\Delta = 0$.

In the above we turn Δ into two determinants (5) and (6) equal to it. The determinant (5) changes to (7) with two alterations of sign, while the determinant (6) changes to $-(7)$ with only one sign change.

With a pan-magic square of order 6,

$$\Delta = \overset{(8)}{\begin{vmatrix} a & b & c & d & e & f \\ g & h & i & j & k & l \\ m & n & o & p & q & r \\ D & E & F & A & B & C \\ J & K & L & G & H & I \\ P & Q & R & M & N & O \end{vmatrix}} = \overset{(9)}{\begin{vmatrix} A & B & C & D & E & F \\ G & H & I & J & K & L \\ M & N & O & P & Q & R \\ d & e & f & a & b & c \\ j & k & l & g & h & i \\ p & q & r & m & n & o \end{vmatrix}}$$

by interchanges of rows and columns. Total of interchanges $=$ an even number.

The determinant (8) is converted into the determinant (10) below—determinant bordered by S's—after three changes of sign.

$$\overset{(10)}{\begin{vmatrix} 6S & S & S & S & S & S \\ S & h & i & j & k & l \\ S & n & o & p & q & r \\ S & e & f & a & b & c \\ S & k & l & g & h & i \\ S & q & r & m & n & o \end{vmatrix}}$$

i.e. replacement of the rows $E, F \ldots$; $K, L \ldots$; $Q, R \ldots$, by their complements $e, f \ldots$; $k, l, \ldots$; $q, r \ldots$. The determinant (9) is changed to (10) after two changes of sign : $H, I \ldots$; $N, O, \ldots$ into $h, i \ldots, n, o \ldots$. Thus the determinant bordered by the S's, *i.e.* $(10) = \pm\Delta$ whence $\Delta = 0$.

In the general case of a pan-magic square of order $2N$, the corresponding determinant is equal to two determinants A and B, where A requires N sign changes to convert it into the final form bordered by S's and free of capital letter quantities while B requires only $N-1$ such changes.

Here again $Z = \pm\Delta$, whence $\Delta = 0$.

We give the 4×4 determinants corresponding to the above for a particular 4×4 pan-magic square.

1	15	6	12
14	4	9	7
11	5	16	2
8	10	3	13

$S = 34$

The determinant of this yields :

$$\Delta = \begin{vmatrix} 4\times 34 & 34 & 34 & 34 \\ 34 & 4 & 9 & 7 \\ 34 & 5 & 16 & 2 \\ 34 & 10 & 3 & 13 \end{vmatrix} = \begin{array}{c} (X) \\ \begin{vmatrix} 4\times 34 & 34 & 34 & 34 \\ 34 & 4 & 9 & 7 \\ 34 & 12 & 1 & 15 \\ 34 & 7 & 14 & 4 \end{vmatrix} \end{array}$$

and
$$\Delta = \begin{vmatrix} 4\times 34 & 34 & 34 & 34 \\ 34 & 13 & 8 & 10 \\ 34 & 12 & 1 & 15 \\ 34 & 7 & 14 & 4 \end{vmatrix} = - \begin{array}{c} (X) \\ \begin{vmatrix} 4\times 34 & 34 & 34 & 34 \\ 34 & 4 & 9 & 7 \\ 34 & 12 & 1 & 15 \\ 34 & 7 & 14 & 4 \end{vmatrix} \end{array}$$

i.e. $\Delta = \pm X$, whence $\Delta = 0$,

X is the determinant bordered by S's.

Summary : We introduce the following general property of pan-magic squares of even order.

Squares of order $2N$ are such that the number of cells along a diagonal, full or broken, separating a quantity a of the square from its complement A, is $N-1$ and the sum of such quantities a and A is $\frac{1}{N}S$.

The above is used to prove that the determinants of even ordered pan-magic squares are all equal to zero.

N. C. and W. J. C.

MATHEMATICAL ASSOCIATION.

ANNUAL MEETING, 1950.

The Annual Meeting will be held at the Polytechnic, Regent Street, London, W. 1, on January 3rd and 4th, 1950. The presidential address will be given by the Astronomer Royal, Sir Harold Spencer Jones, F.R.S.

The lecturers will include Professor D. R. Hartree, Professor H. S. W. Massey and Dr. F. Smithies, and there will be two discussions, one on the Trigonometry Report and the other on Mathematics in the Comprehensive School.

The successful hostel arrangements at Birmingham have prompted members to ask if hostel accommodation could be provided when the Annual Meeting is held in London. The Programme Committee is glad to announce that accommodation can be arranged from 2nd to 5th January, 1950, at College Hall (University of London), Malet Street, W.C. 1, provided *at least forty* members undertake to stay there. The charge is 15s. per day for dinner, bed and breakfast. It is very important that members wishing to take advantage of such an arrangement should notify Dr. J. Topping, The Polytechnic, Regent Street, W. 1, *as soon as possible*.

SQUARE ROOTS OF INTEGERS EXPRESSED AS INFINITE SERIES.

BY G. L. CAMM.

It is well known that square roots of integers and rational numbers can be calculated by means of recurrence relations. Further, the same recurrence relations can sometimes be used to express a square root as a continued fraction. From a certain type of recurrence relation which gives a square root it is also possible to express that root as a rapidly convergent infinite series. As might be expected, the method is connected with the theory of continued fractions, and that connection will be given.

The method of recurrence relations for surds of the form $\sqrt{(m^2-1)}$.

It is convenient to give first a derivation which is a particular case, but which illustrates the general method. We consider the quadratic equation,

$$x^2 - 2x\cosh\theta + 1 = 0, \quad \theta > 0, \quad \text{.........................(1)}$$

and the set of recurrence formulae,

$$a_{n+2} - 2a_{n+1}\cosh\theta + a_n = 0, \quad n \geqslant 0. \quad \text{.......................(2)}$$

The roots of (1) are $x = e^{\theta}$ or $e^{-\theta}$, and the general solution of (2) is

$$a_n = Ae^{n\theta} + Be^{-n\theta},$$

where A and B are arbitrary constants. If we impose on (2) the initial conditions $a_0 = 1$, $a_1 = e^{-\theta}$, then $A = 0$, $B = 1$, and $a_n = e^{-n\theta}$. Clearly a_n tends to zero as n tends to infinity.

But a_1 can be expressed linearly as a function of the other a's. We have

$$a_1 = \frac{a_0}{2\cosh\theta} + \frac{a_2}{2\cosh\theta}.$$

Also we have, either by eliminating a_1 and a_3 from the first three recurrence relations, or else by inspection, the equation $a_0 - 2a_2\cosh 2\theta + a_4 = 0$. It follows that

$$a_1 = \frac{a_0}{2\cosh\theta} + \frac{a_0}{(2\cosh\theta)(2\cosh 2\theta)} + \frac{a_4}{(2\cosh\theta)(2\cosh 2\theta)}.$$

More generally, we can write

$$a_1 = \frac{a_0}{2\cosh\theta} + \ldots + \frac{a_0}{\prod\limits_{r=0}^{n}(2\cosh 2^r\theta)} + \frac{a_{2^{n+1}}}{\prod\limits_{r=0}^{n}(2\cosh 2^r\theta)}.$$

This is readily proved by induction, using the relation

$$a_0 - 2a_{2^n}\cosh 2^n\theta + a_{2^{n+1}} = 0.$$

Thus we have

$$a_1 - a_0\sum_{n=0}^{p}\left\{\prod_{r=0}^{n}(2\cosh 2^r\theta)\right\}^{-1} = a_{2^{p+1}}\left\{\prod_{r=0}^{p}(2\cosh 2^r\theta)\right\}^{-1}.$$

Putting $a_0 = 1$, $a_1 = e^{-\theta}$, we can write this in the form:

$$\begin{aligned} e^{-\theta} - \sum_{n=0}^{p}\left\{\prod_{r=0}^{n}(2\cosh 2^r\theta)\right\}^{-1} &= e^{-2^{p+1}\theta}\left\{\prod_{r=0}^{p}(2\cosh 2^r\theta)\right\}^{-1} \\ &= \{e^{-2^{p+1}\theta}\sinh\theta\}/\sinh 2^{p+1}\theta \\ &= \frac{e^{\theta}-e^{-\theta}}{e^{2^{p+2}\theta}-1}. \end{aligned}$$

Since the right-hand side tends to zero as p tends to infinity, we have that $e^{-\theta}$ is the sum of the infinite series,

$$\sum_{n=0}^{\infty}\left\{\prod_{r=0}^{n}(2\cosh 2^r\theta)\right\}^{-1}.$$

Now suppose that $m(>1)$ is a rational number ; the equation

$$x^2-2mx+1=0$$

has roots, $x=m\pm\sqrt{(m^2-1)}$. If $\cosh\theta=m$, then $e^{-\theta}=m-\sqrt{(m^2-1)}$. Thus we can write

$$\sqrt{(m^2-1)}=m-\sum_{n=0}^{\infty}\left\{\prod_{r=0}^{n}(2\cosh 2^r\theta)\right\}^{-1}.$$

Denoting $2\cosh 2^r\theta$ by q_r, we have the relation

$$\begin{aligned} q_r&=\exp(2^r\theta)+\exp(-2^r\theta)\\ &=\{\exp(2^{r-1}\theta)+\exp(-2^{r-1}\theta)\}^2-2\\ &=q_{r-1}^2-2.\end{aligned}$$

Also $\qquad q_0=2m.$

The series can now be written in the form

$$\sqrt{(m^2-1)}=m-\sum_{n=0}^{\infty}\left\{\prod_{r=0}^{n}q_r^{-1}\right\}, \quad\text{.........................(3)}$$

or

$$\sqrt{(m^2-1)}=m-\frac{1}{q_0}\left[1+\frac{1}{q_1}\left[1+\frac{1}{q_2}\left[1+\dots\right]\right]\right].$$

In terms of m, the first few terms are

$$\sqrt{(m^2-1)}=m-\frac{1}{2m}\left[1+\frac{1}{4m^2-2}\left[1+\frac{1}{16m^4-16m^2+2}\left[1+\dots\right]\right]\right]. \quad\text{.....(4)}$$

Surds of the form $\sqrt{(m^2+1)}$.

By considering the equation

$$x^2+2x\sinh\theta-1=0,\quad \theta>0,$$

we can show that

$$e^{-\theta}=\frac{1}{2\sinh\theta}-\frac{1}{2\sinh\theta}\sum_{n=1}^{\infty}\left\{\prod_{r=1}^{n}(2\cosh 2^r\theta)\right\}^{-1}.$$

Now put $\qquad m=\sinh\theta,\quad \sqrt{(m^2+1)}=\cosh\theta.$

Then we have

$$\sqrt{(m^2+1)}=m+\frac{1}{2m}-\frac{1}{2m}\sum_{n=1}^{\infty}\left\{\prod_{r=1}^{n}(2\cosh 2^r\theta)\right\}^{-1}.$$

Defining quantities q by the equations

$$\begin{aligned} q_0&=2m,\\ q_1&=q_0^2+2,\\ q_{r+1}&=q_r^2-2,\quad r\geqslant 1,\end{aligned}$$

we can write this in the form

$$\sqrt{(m^2+1)}=m+\frac{1}{q_0}\left[1-\frac{1}{q_1}\left[1+\frac{1}{q_2}\left[1+\dots\right]\right]\right]. \quad\text{..................(5)}$$

In terms of m, the first few terms are

$$\sqrt{(m^2+1)}=m+\frac{1}{2m}\left[1-\frac{1}{4m^2+2}\left[\ +\frac{1}{16m^4+16m^2+2}\left[1+\dots\right]\right]\right]. \quad\text{.....(6)}$$

More general series.

By similar methods we can obtain a series for $\sqrt{(m^2-n)}$, $0<n<2m-1$. If we define two sequences p_r, q_r by the relations

$$p_{r+1}=p_r^2, \quad p_1=n, \quad q_{r+1}=q_r^2-2p_r, \quad q_1=2m,$$

then we obtain the result,

$$\sqrt{(m^2-n)}=m-\sum_{t=1}^{\infty}\left\{p_t\prod_{r=1}^{t}(q_r^{-1})\right\}. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

Finally, if sequences p_r, q_r are defined by the relations

$$p_{r+1}=p_r^2, \quad p_1=n^2, \quad q_{r+1}=q_r^2-2p_r, \quad q_1=4m^2+2n,$$

then for $0<n<2m+1$ we have

$$\sqrt{(m^2+n)}=m+\frac{n}{2m}-\frac{1}{2m}\sum_{t=1}^{\infty}\left\{p_t\prod_{r=1}^{t}(q_r)^{-1}\right\}. \quad\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

In all cases it will be seen that the first two terms of the series coincide with the binomial expansion. However, the series converges much more rapidly than the binomial series. The terms after the second are all of the same sign, and they decrease much more rapidly. In the binomial series the ratio of successive terms is of order n/m^2. In the series (7) and (8) the ratio is p_{t+1}/p_tq_{t+1}, or $p_t/(q_t^2-2p_t)$. Now we have the inequalities,

$$\begin{aligned}
&q_r^2>q_{r+1}>q_r^2-4p_r,\\
&q_{r-1}^4>q_{r+1}>q_r^2-4p_{r-1}^2,\\
&q_{r-2}^8>q_{r+1}>(q_r-2p_{r-1})(q_r+2p_{r-1}),\\
&q_{r-3}^{16}>q_{r+1}>(q_{r-1}^2-4p_{r-1})q_{r-1}^2,\\
&q_1^{2^r}>q_{r+1}>(q_1^2-4p_1)q_{r-1}^2q_{r-2}^2\ldots q_2^2q_1^2.
\end{aligned}$$

By the repeated use of the right-hand half of this inequality it is found that:

$$\begin{aligned}
&q_{2r}>(q_1^2-4p_1)^{\frac{1}{3}(2^{2r-1}+1)}q_1^{\frac{1}{3}(2^{2r-1}-2)},\\
&q_{2r+1}>(q_1^2-4p_1)^{\frac{1}{3}(2^{2r}-1)}q_1^{\frac{1}{3}(2^{2r}+2)}.
\end{aligned}$$

It now follows that the ratio of successive terms, p_{2r}/q_{2r+1}, must be less than

$$p_1^{2^{2r-2}}/(q_1^2-4p_1)^{\frac{1}{3}(2^{2r}-1)}q_1^{\frac{1}{3}(2^{2r}+2)}.$$

In equation (8), for example, this ratio is certainly less than $n^{2^{2r}}/(2m)^{2^{2r+1}}$.

For purposes of computation the series (7) and (8) are rather inconvenient to use—unless the first few terms are sufficient for the approximation required. To obtain each term from the preceding one, a multiplication is needed as well as a division. The case of $n=4$ is exceptional in that the numerators all cancel. We then have the series:

$$\sqrt{(m^2-4)}=m-\frac{2}{m}\left[1+\frac{1}{m^2-2}\left[1+\frac{1}{m^4-4m^2+2}\left[1+\ldots\right]\right]\right],$$

$$\sqrt{(m^2+4)}=m+\frac{2}{m}\left[1-\frac{1}{m^2+2}\left[1+\frac{1}{m^4+4m^2+2}\left[1+\ldots\right]\right]\right].$$

It will now be shown that the square root of any rational number can be found from one or other of the series (3) and (5). This means that each term of the series can be obtained by dividing the previous term, so that the process is very convenient for the use of a calculating machine. The modification depends on some properties of continued fractions, which will now be considered.

Square roots expressed as continued fractions.

It is well known * that if D is a positive integer, which is not itself a square, then its square root can always be represented by a periodic continued fraction, so that

$$\sqrt{D} = a + \frac{1}{\alpha_1 +}\,\frac{1}{\alpha_2 +}\cdots\frac{1}{\alpha_{n-1} +}\,\frac{1}{2a +}\,\frac{1}{\alpha_1 +}\cdots .$$

In this, a and the $(n-1)$ quantities α are integers, and the n elements

$$\left(\frac{1}{\alpha_1 +}\,\frac{1}{\alpha_2 +}\cdots\frac{1}{\alpha_{n-1} +}\,\frac{1}{2a}\right)$$

are recurrent. Now denote by x_m/y_m the (mn)th convergent of this infinite continued fraction, so that x_m/y_m is the value of the *finite* continued fraction obtained by ending the fraction after the mth appearance of α_{n-1}. Then the integers x_m and y_m satisfy the Pellian equation,

$$x_m^2 - Dy_m^2 = (-1)^{mn}. \qquad \text{.......................(9)}$$

From this it follows that we can write

$$\sqrt{D} = \frac{1}{y_m}\sqrt{(x_m^2 \pm 1)}.$$

The square root on the right-hand side can be expanded in one or other of the series (3) and (5). In the case of a rational number p/q, we find the solution of the equation

$$x_m^2 - pqy_m^2 = \mp 1,$$

(by expanding $\sqrt{(pq)}$ as a continued fraction), and hence we have

$$\sqrt{\left(\frac{p}{q}\right)} = \frac{1}{q}\sqrt{(pq)} = \frac{1}{qy_m}\sqrt{(x_m^2 \pm 1)}.$$

This completes the formal solution of the problem, the expression of the square root of a rational number as a rapidly convergent infinite series.

The relation between the infinite series and the continued fraction.

The ratios x_m/y_m (or $x(m)/y(m)$) are a subset of all the convergents of $\sqrt{D}$ expressed as a continued fraction, and tend to $\sqrt{D}$ as m tends to infinity. We can now show that there is a subset of the ratios x_m/y_m which is identical with the partial sums of the series for $\sqrt{D}$ obtained in the manner just described. This depends on the identities † between $x(2m)$, $y(2m)$ $x(m)$ and $y(m)$, namely

$$x(2m) = x^2(m) + Dy^2(m),$$
$$y(2m) = 2x(m)y(m).$$

The theorem will be stated and proved for a pair of numbers $x(m)$ and $y(m)$ which satisfy

$$x^2(m) - Dy^2(m) = 1. \qquad \text{.......................(10)}$$

There is clearly a comparable theorem when $x(m)$, $y(m)$ satisfy the relation

$$x^2(m) - Dy^2(m) = -1,$$

but from equation (9), it will be seen that (10) is the more usual form. If $x(m)$, $y(m)$ satisfy (9), then we have

$$\sqrt{D} = \frac{1}{y(m)}\left[x(m) - \sum_{p=0}^{\infty}\left\{\prod_{t=0}^{p}(q_t)^{-1}\right\}\right], \qquad \text{.......................(11)}$$

* Cf. Chrystal, *Algebra*, XXXIII, p. 435.

† Chrystal, *Algebra*, XXXIII, p. 440.

where $q_0 = 2x(m)$, $q_{t+1} = q_t^2 - 2$. Let us define $S(r)$ by the equation,

$$S(r) = \frac{1}{y(m)}\left[x(m) - \sum_{p=0}^{r-1}\left\{\prod_{t=0}^{p}(q_t)^{-1}\right\}\right];$$

then it will be shown that

$$S(r) = x(2^{r-1}m)/y(2^{r-1}m). \qquad (12)$$

The theorem is obviously true for $r = 1$. Similarly, we have

$$\begin{aligned} S(2) &= \frac{1}{y(m)}\left[x(m) - \frac{1}{2x(m)}\right] \\ &= \frac{1}{2x(m)y(m)}[2x^2(m) - 1] \\ &= \frac{1}{2x(m)y(m)}[x^2(m) + Dy^2(m)] \\ &= \frac{x(2m)}{y(2m)}. \end{aligned}$$

Thus the theorem is also true for $r = 2$. Suppose it is true for all integers up to $r = p$.

Then it follows that for $2 \leqslant n \leqslant p$,

$$S(n) - S(n-1) = \frac{x(2^{n-1}m)}{y(2^{n-1}m)} - \frac{x(2^{n-2}m)}{y(2^{n-2}m)}.$$

That is,

$$\begin{aligned} -\frac{1}{y(m)}\prod_{t=0}^{n-1}(q_t)^{-1} &= \frac{x(2^{n-1}m)}{y(2^{n-1}m)} - \frac{x(2^{n-2}m)}{y(2^{n-2}m)} \\ &= \frac{x^2(2^{n-2}m) + Dy^2(2^{n-2}m) - 2x^2(2^{n-2}m)}{2x(2^{n-2}m)y(2^{n-2}m)} \\ &= \frac{-1}{y(2^{n-1}m)}. \end{aligned}$$

Hence it follows that, for $2 \leqslant n \leqslant p$,

$$y(m)\prod_{t=0}^{n-1}(q_t) = y(2^{n-1}m).$$

Dividing this by $y(m)\prod_{t=0}^{n-2}(q_t) = y(2^{n-2}m)$, we obtain

$$\begin{aligned} q_{n-1} &= y(2^{n-1}m)/y(2^{n-2}m) \\ &= 2x(2^{n-2}m). \end{aligned}$$

In particular, we have

$$q_{p-1} = 2x(2^{p-2}m).$$

But

$$\begin{aligned} q_p &= q_{p-1}^2 - 2 \\ &= 4x^2(2^{p-2}m) - 2 \\ &= 2x(2^{p-1}m). \end{aligned}$$

It follows that

$$\begin{aligned} y(m)\prod_{t=0}^{p}(q_t) &= y(2^{p-1}m)\,.\,2x(2^{p-1}m) \\ &= y(2^p m). \end{aligned}$$

Hence
$$S(p+1)-S(p) = -\frac{1}{y(m)}\prod_{t=0}^{p}(q_t)^{-1}$$
$$= -\frac{1}{y(2^p m)}$$
$$= \frac{Dy^2(2^{p-1}m)-x^2(2^{p-1}m)}{y(2^p m)}$$
$$= \frac{x(2^p m)-2x^2(2^{p-1}m)}{y(2^p m)}$$
$$= \frac{x(2^p m)}{y(2^p m)} - \frac{x(2^{p-1}m)}{y(2^{p-1}m)}.$$

But
$$S(p) = x(2^{p-1}m)/y(2^{p-1}m),$$
so that
$$S(p+1) = x(2^p m)/y(2^p m).$$

Hence, if (12) is true for all $r \leqslant p$, it is true for $r = p+1$. But it is true for $r = 1$, 2, so it is true for all r.

It is this property which accounts for the rapid convergence of the series (11). If $\sqrt{D}$ has n repeated elements when expressed as a continued fraction, then $x(m)/y(m)$ is the mnth convergent, and the sum of r terms of (11) is the $(2^{r-1}mn)$th convergent. For a given D the period n is fixed, but both m and r are at our disposal. For the rapid convergence of (11) it is necessary that $x(m)$ should be large. If $x(1)$ is not sufficiently large, we can find larger values, $x(m)$, $y(m)$ from the formulae :

$$x(m+r) = x(m)x(r) + Dy(m)y(r),$$
$$y(m+r) = x(m)y(r) + y(m)x(r).$$

The smallest value of $x(1)$ is 2, arising from the surd $\sqrt{3}$. This value makes (11) converge most slowly. Seven terms are needed to calculate $\sqrt{3}$ to 50 places of decimals. The next smallest solution of the equation

$$x^2 - 3y^2 = 1$$

is $x(2) = 7$, $y(2) = 4$. This gives the same series as $x(1) = 2$, $y(1) = 1$, but combines the first two terms. The next solution, $x(3) = 26$, $y(3) = 15$, gives an entirely different series, of which five terms are needed for 50 places. The solution, $x(4) = 97$, $y(4) = 56$, reproduces the series obtained from $x(1)$, $y(1)$, but the first three terms are combined. The solution, $x(5) = 362$, $y(5) = 209$ gives yet a third series, but still requires five terms. More generally, the solution, $x(2m+1)$, $y(2m+1)$, determines a series for $\sqrt{D}$ which is fundamentally different from the series obtained from a convergent of lower order. The solution $x(2m)$, $y(2m)$ gives the same series as $x(m)$, $y(m)$, but combines the first two terms into a single term.

The more usual result is not that $x(m)$, $y(m)$ are too small, but that they are extremely large.* If $x(m)$ is of order 10^p, and $y(m)$ of order 10^q, $(p, q > 2)$, then the second term is of order $10^{-(p+q)}$, the third $10^{-(3p+q)}$, and the fourth $10^{-(7p+q)}$. It follows that the sum of the first three terms will express the square root accurately to 50 places if $7p + q > 50$, that is, since $p \geqslant q$, $p \geqslant 7$. Thus the large values obtained as solutions of Pell's equation have the advantage that only three terms of the series need to be evaluated.

It is a pleasure to acknowledge the many helpful suggestions made by Mr. D. F. Ferguson, with whom these results have been discussed. G. L. C.

* Solutions of Pell's equation, $x^2 - Dy^2 = \pm 1$, have been tabulated for all integers D up to 2000. See E. L. Ince, *Cycles of Reduced Ideals in Quadratic Fields*, B.A. Tables, 1934.

ON PEDAL AND ANTIPEDAL TRIANGLES.

By R. Goormaghtigh.

1. It is well known that the Lemoine point of a triangle ABC is the centroid of its pedal triangle with respect to ABC.

This suggests to find a point P which is the centroid of its antipedal triangle as to ABC.

Let LMN be the antipedal triangle of P and A', B', C' the inverse of A, B, C with respect to a circle Γ having its centre at P. If J is the intersection of AP and $B'C'$, the inverse J' of J as to Γ is the point where the parallel to MN drawn through L meets again the circle $BPCL$. Hence the ratio $PJ : A'J$ equals $PA : J'A$, and this means that the barycentric coordinates of P are the same in the triangle LMN as in the triangle $A'B'C'$. Therefore, if P is the centroid of its antipedal triangle, it is also the centroid of $A'B'C'$.

We will now use complex coordinates, the origin being any point O and the unit circle being taken equal to Γ. Let α, β, γ be the coordinates of A, B, C, and denote by $\bar{z}$ the conjugate to z and by σ_1 and σ_2 the sums

$$\sigma_1 = \alpha + \beta + \gamma, \quad \sigma_2 = \beta\gamma + \gamma\alpha + \alpha\beta.$$

Then the coordinate of the inverse of a point z as to the unit circle is $1/\bar{z}$, and the point P, having x for coordinate, will be the centroid of $A'B'C'$ when

$$(\bar{x} - \bar{\alpha})^{-1} + (\bar{x} - \bar{\beta})^{-1} + (\bar{x} - \bar{\gamma})^{-1} = 0,$$

i.e. when

$$3x^2 - 2\sigma_1 x + \sigma_2 = 0. \quad \text{.................................(1)}$$

So there are two points P_1 and P_2 having the considered property; the midpoint of their distance has for coordinate $\sigma_1/3$ and is the centroid G of ABC.

If x_1 and x_2 are the roots of (1), *i.e.* the coordinates of P_1 and P_2, we have the identity

$$3(x - x_1)(x - x_2) = \Sigma(x - \beta)(x - \gamma)$$

and its conjugate

$$3(\bar{x} - \bar{x}_1)(\bar{x} - \bar{x}_2) = \Sigma(\bar{x} - \bar{\beta})(\bar{x} - \bar{\gamma}).$$

When $x = \alpha$,

$$3(\alpha - x_1)(\alpha - x_2) = (\alpha - \beta)(\alpha - \gamma),$$

$$3(\bar{\alpha} - \bar{x}_1)(\bar{\alpha} - \bar{x}_2) = (\bar{\alpha} - \bar{\beta})(\bar{\alpha} - \bar{\gamma})$$

and

$$\frac{\alpha - x_1}{\bar{\alpha} - \bar{x}_1} \cdot \frac{\alpha - x_2}{\bar{\alpha} - \bar{x}_2} = \frac{\alpha - \beta}{\bar{\alpha} - \bar{\beta}} \cdot \frac{\alpha - \gamma}{\bar{\alpha} - \bar{\gamma}}. \quad \text{.........................(2)}$$

But, if θ is the angle formed by the coordinates-axis and the join of the points u, v,

$$(u - v)/(\bar{u} - \bar{v}) = e^{i\theta}.$$

Hence (2) means that the angles P_1AP_2 and BAC have the same bisectrices and P_1 and P_2 are the foci of the conic inscribed in the triangle ABC and having G for centre, *i.e.* the inscribed Steiner ellipse.

In a triangle there are two points which are the centroids of their respective antipedal triangles; these are the foci of the inscribed Steiner ellipse.

2. The above recalled property of the Lemoine point may be easily generalised.

Let a, b, c be the lengths of the sides of ABC. A point having p, q, r for normal coordinates in the triangle ABC has a/p, b/q, c/r for barycentric coordinates in its pedal triangle; hence, the point which has, in its pedal

triangle, for barycentric coordinates λ, μ, ν is that having a/λ_1, b/μ_1, c/ν for normal coordinates in the triangle ABC.

3. We will now generalise the property about antipedal triangles and find the points which have given numbers λ, μ, ν for absolute barycentric coordinates in their antipedal triangles, the sum $\lambda+\mu+\nu$ being unit.

If y is the complex coordinate of such a point, we must have, according to § 1,

$$\lambda(\bar{y}-\bar{\alpha})^{-1}+\mu(\bar{y}-\bar{\beta})^{-1}+\nu(\bar{y}-\bar{\gamma})^{-1}=0,$$

or

$$y^2-y\Sigma\lambda(\beta+\gamma)+\Sigma\lambda\beta\gamma=0,$$

or

$$y^2-y(\sigma_1-\Sigma\lambda\alpha)+\Sigma\lambda\beta\gamma=0. \quad\text{.........................(3)}$$

There are generally two points Q_1 and Q_2 having the considered property, and the midpoint of their distance is the point

$$(\sigma_1-\Sigma\lambda\alpha)/2. \quad\text{.................................(4)}$$

But $\Sigma\lambda\alpha$ is the complex coordinate of the point M having λ, μ, ν for barycentric coordinates in the triangle ABC and (4) is the complex coordinate of the point M' such that $\overline{MG}=2\overline{GM'}$, *i.e.* the complementary point to M as to the triangle ABC.

Further, a relation similar to (2) will be easily derived from (3), and Q_1 and Q_2 are also the foci of a conic inscribed in ABC.

In a triangle there are two points which have given numbers λ, μ, ν for barycentric coordinates in their respective antipedal triangles ; these are the foci of the inscribed conic having for centre the complementary point to that whose barycentric coordinates in the triangle ABC are λ, μ, ν.

4. There will be only one point Q when M is the anti-complementary point to the centre of one of the tritangent circles, *i.e.* the Nagel point or one of its associated points. If $2s$ denotes the sum $a+b+c$, the corresponding values of λ, μ, ν are then proportional to

$$(s-a,\ s-b,\ s-c),\qquad(-s,\ s-c,\ s-b),$$
$$(s-c,\ -s,\ s-a),\qquad(s-b,\ s-a,\ -s).$$

It is easy to verify that, for instance, the barycentric coordinates of the in-centre in the triangle formed by the ex-centres are as $s-a$, $s-b$, $s-c$, as the first point is the orthocentre of the triangle formed by the three others, and therefore its barycentric coordinates in this last triangle are as the tangents of the angles of that triangle, *i.e.* as $\cot A/2$, $\cot B/2$, $\cot C/2$, A, B, C being the angles of ABC.

5. Equation (3) leads also to an interesting property of the points Q_1 and Q_2.

The inverse points A_1, B_1, C_1 of A, B, C with respect to the unit-circle have for complex coordinates $1/\bar{\alpha}$, $1/\bar{\beta}$, $1/\bar{\gamma}$, and the point T having in the triangle $A_1B_1C_1$ for barycentric coordinates λ, μ, ν is

$$\xi=(\Sigma\lambda\bar{\beta}\bar{\gamma})/\bar{\alpha}\bar{\beta}\bar{\gamma}.$$

Hence

$$\overline{OQ}_1{}^2\,.\,\overline{OQ}_2{}^2=\Sigma\lambda\beta\gamma\Sigma\lambda\bar{\beta}\bar{\gamma}=\alpha\bar{\alpha}\beta\bar{\beta}\gamma\bar{\gamma}\xi\bar{\xi}=\overline{OA}^2\,.\,\overline{OB}^2\,.\,\overline{OC}^2\,.\,\overline{OT}^2.$$

If A_1, B_1, C_1 are the inverse of A, B, C as to a circle having for centre any point O and for radius ρ, and if T is the point having λ, μ, ν for barycentric coordinates in the triangle $A_1B_1C_1$, then

$$\rho^2\,.\,OQ_1\,.\,OQ_2=OA\,.\,OB\,.\,OC\,.\,OT.$$

This may be considered as an extension of Aiyar's theorem on counterpoints, as the foregoing theorem may also be stated in the following form :

If Q_0 is the midpoint of the distance between two counterpoints Q_1 and Q_2, if A_1, B_1, C_1 are the inverse points of A, B, C as to any circle having for centre O and for radius ρ, and if T is the point which has, in the triangle $A_1B_1C_1$, the same barycentric coordinates as the anticomplementary to Q_0 in the triangle ABC, then

$$\rho^2 . OQ_1 . OQ_2 = OA . OB . OC . OT.$$

When the considered circle is the circle ABC, having R for radius, A_1, B_1, C_1 coincide with A, B, C respectively and the distance OT is twice that from the nine-point centre O_9 to Q_0; hence we find Aiyar's relation *

$$OQ_1 . OQ_2 = 2R . O_9Q_0.$$

R. G.

* Gallatly, *Modern Geometry of the Triangle*, 2nd edition, p. 79.

GLEANINGS FAR AND NEAR.

1596. The ratio between the load (L) (i.e. the total pressure between the surfaces) and the effort (E) you have to make to get the surfaces moving past each other, is the measure of the force (F) you have to use.

The scientific name for this force is the coefficient of friction ; and the act can be expressed in the form : F equals E/L.—*Safety Training*, Vol 10, No. 54, 1948, p. 8. [Per Dr. L. John Stroud.]

1597. When I planned the farm expenditure in June, 1946, prices stood, let us say, at £x. When in November I took the farm and began to buy, prices were already standing at £$x + \frac{1}{4}$ to $\frac{1}{3}$; now, in May 1947, prices are £$x + \frac{1}{3}$ to $\frac{1}{2}$, all of which means that the things that I have had to buy . . . are all from one-third to one-half as much again as I bargained for.—C. E. M. Joad, *A Year More or Less* (1948), p. 182. [Per Prof. E. H. Neville.]

1598. If the strength of the pack were divided evenly each player would have 10 points. With no indication to the contrary, partner must be assumed to hold an average. So if you yourself have 2 or 3 points more than your share, your combined holding will be 22–23.—Victor Mollo, *Streamlined Bridge* (1947), p. 15. [Per Prof. E. H. Neville.]

1599. It is true that he depreciated pure mathematics ; but there are at least some eminent philosophers of today who would enthusiastically approve his relegation of the science to a purely ancillary role. He saw quite correctly the promising field physics offers to " mixed ", or as a modern world would say " applied mathematics ". This is the more remarkable, since he was ignorant of Kepler's work, which was being published at the time.—H. C. O'Neill, " Francis Bacon ", *The Great Tudors* (1935, ed. K. Garvin), p. 638.

1600. Lyly's peculiar mode of expression is not idiom but formula. No writer in the history of English literature has so reduced the art of writing to a mathematical canon. It is machine-made throughout, and has actually been tabulated in algebraical terms—" as the A is B, so the C is D and the more E is F, the more G is H ". This series of balanced antithesis, oiled and polished by alliteration's artful aid, works with the precision of a steam-hammer, so that you can time the beats and calculate the stresses almost as accurately as in a factory.—H. J. Massingham, " John Lyly ", *The Great Tudors* (1935, ed. K. Garvin), p. 572.

MATHEMATICS AND THE CHILD. II.

By C. Gattegno.

In the first part of this article we studied the relation between mathematics and the child and tried to suggest that a greater place could be given to mathematics in the schools by taking into account how the child learns. We tried to show that, to give the child a knowledge of mathematics, various approaches can be used, each of which can serve as a basis for the following one if we follow the development of logic, using the set of tools mastered at each stage in order to progress from one level to the next.

Before we embark on our second part, two points require to be clarified.

1. We spoke of substitutes for mathematical facts at various levels without perhaps giving an adequate explanation of what we mean by a substitute.

To show pupils that in a triangle $\angle A + \angle B + \angle C = 180°$ the method most commonly used is to ask them to add up the measurements of the three angles. For us, $\angle A + \angle B + \angle C = 179°$ is not a substitute for the theorem. It is an approximate result if the true result is known, otherwise it is a particular finding resulting from certain actions of the child.

Substitutes are mathematical truths obtained otherwise than by reasoning. In spite of being temporary expedients, they are quite as efficient for the child as a mathematical understanding is for us. $\angle A + \angle B + \angle C = 179°$ being wrong mathematically is not a substitute. But logarithm tables correctly used, though without understanding either of their origin or construction, are a substitute. If in order to solve $\frac{a}{b} = \frac{1}{x}$ I suggest reversing the fractions and obtain $\frac{x}{1} = \frac{b}{a}$ I am giving a substitute, an action as substitute for the solution making full use of the rules.

These examples will, we hope, make clear what is meant by a substitute.

2. The second point concerns the idea of abstraction, a word used very loosely by many psychologists and whose meaning is assumed to be quite clear. Obviously there are hierarchies of abstractions such as 2 and x and $\int \frac{dx}{\log x}$ and $\aleph_0$, etc. It is also evident that animals and even plants have some capacity for achieving analytic concepts of the kind that we are continually manipulating. Are the selective powers of a tissue anything else than *the ability to distinguish between like and unlike*? The point we wish to make is that it is not only *verbal* possibilities of distinction which must be taken into account when there is some truth to be grasped. Analytic powers work very early in the mind of human beings, but in a way not usually appreciated or recognised by the adult mind. The reason for our failure to recognise this fact is to be found in an ordinary experience of the human mind, the construction of ever-widening classes of attributes, each new one containing and taking the place of the previous one. For instance, the class of red objects is contained in the class of coloured objects. To grasp the first of these, we needed a certain amount of experience and some power of abstraction; to grasp the second, a conception of various classes each of a definite colour was necessary. Red becomes a colour only when it is dissociated from the objects which have this quality. When it becomes a colour it is an abstract red and can be used in combinations of colour.

This somewhat detailed explanation has been necessary to make us aware of the value of substitutes in the teaching of mathematics. Red does not lose its value as a vibration of the ether when it is used and investigated by painters merely as a colour acting on the retina and combined with other

colours. The same can be said of what we are proposing here. We do not lose the mathematical value of any of the propositions we present to the child if we present them as containing only those "dimensions" which the child can grasp. The process of his learning will be gradually to add more and more "dimensions" to the idea already grasped.

(In another paper, "Learning by Mistakes", we develop this point.)

The giving of geometrical toys to children has long been advocated, and many are used in Dr. Montessori's method. Though this idea has been offered to educationists for more than fifty years, very little has in practice been done with it, and there is great opportunity for improvement and extension. (We hope one day to show how much can be done in this respect by a single contributor, and to stimulate others to contribute.)

The well-known "insets" certainly provide a true mathematical acquaintance with figures and relationships of figures. My daughter at the age of three and a half was able in a Montessori class to distinguish clearly between the centre of an area and the middle of a line, and would not allow me to confuse these two concepts in ordinary speech.

Apart from this material for early stages, very little exists that can be used by children for further development, partly because the only mathematics taught in primary schools is arithmetic, and partly because no one has been sufficiently interested to produce active material corresponding to the ages 7 to 12. Our ancestors used simple machines for counting which have been easily replaced by mental arithmetic, and it has seemed that there was no real need for new devices in this part of the mathematics syllabus. I cannot accept this view, but a description of some substitutes for arithmetic would take me too far.

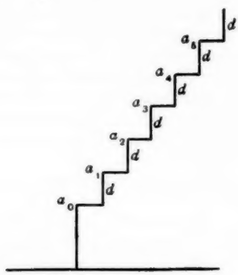
After the age of 11+ children are suddenly brought into contact with algebra and geometry, whose nature is quite alien to their previous experience and their present interests. What they really know and can do is connected with sensorial and motorial abstractions. For instance, at that age $2+3=5$ is a motorial operation; the sign $+$ gives a dynamical order. Then suddenly they are faced with $x+y$, which is static, and they fail to understand it, since there is really nothing there which can be understood at their level. $x+y$ is always and for everyone $x+y$. What is new in it is quite outside their sensorial and active power of analysis. They cannot put x and y together unless they write xy, as they often do, remembering the fusion of 2 and 3 into 5, the symbol thus replacing the result. They cannot make the addition as they have been trained for years to do, they cannot conceive of x and y as anything but two fixed letters or objects. Very late in their school life they still react in this way, because algebra has been kept outside their real life, which is sensorial, active, emotional before being freely intellectual.

My approach to *the creation of substitutes is to give a sensorial, active, emotional value to the intellectual process*. I make small machines to perform mechanical processes and I accept a process as being understood if the right result is obtained following mechanical and not verbal-mental lines. For example, I ask pupils to factorise $ax+bx$ with their eyes, and to show them how to do it I cover x with my hands, leaving visible the other factor, $a+b$. I do not remind the pupils that we multiply a and x, b and x, and add them, or add them and then multiply by x. For me what is involved is an action which the child can perform and which gives the required result. Later on in their school life the pupils will see the full meaning of the process, as they do that of so many others which we never explain. Presented like this, factorisation of expressions like $ax+bx+ay+by$ follows immediately, and the existence of a minus sign does not introduce any new difficulty. The method has been completely successful.

What are the parts of algebra that can be dealt with by mechanical processes? Obviously factorisation, simplification of fractions, L.C.M., H.C.F., arithmetical and geometrical progressions are of this type.

Let us take the example of arithmetical progressions. I make a staircase starting from any height a_0 and call d the height of a step. I put, say, a rabbit on the floor, make it jump, and calculate the heights it reaches. If we start from any step instead of from the floor, we can use the same process and let the pupils discover that $a_n = a_0 + nd$ or $a_q = a_p + (q-p)d$. Then we look at the figure and see that the first height a_0 may vary, as also may d and the step on which we stop. The final height reached will depend on these three data. A certain amount of play is possible here, and is indeed inevitable if the picture is to become familiar. We then make holes in the steps, produce the floor to a wall, and have two people to shout and listen to the echo at various stages, and we find that the time for a return journey of the sound is such that at two steps equidistant from the ends it adds up to the same.

$a_{n-p} + a_p$ is independent of p and therefore $= a_n + a_0$.



$$a_n = a_0 + nd, \quad a_q = a_p + (q-p)d, \quad a_{n-p} + a_p = a_n - a_0, \quad 2S = (n+1)(a_0 + a_n).$$

If we now send one rabbit up from the floor and another down from the top (a_n), both jumping at the same speed, and make them stop on each step to discover the height they have reached, we find that both have obtained the same numbers, but in reverse order, and that the numbers add up to the same result. The formula we obtain is to some extent artificial for the pupil, but each operation can be visualised and the difference between a_{n-1} and a_n appreciated, a rather difficult achievement by some other methods. We have tried this method with success several times. The gestures necessary to complete this description are left for the reader to supply. Many other methods are of course possible, and from what we have said it is obvious that a film could easily be made to show what an arithmetical progression is. For geometrical progressions the staircase would be replaced by a sequence of wheels magnifying the numbers.

I would ask the reader not to criticise these methods on philosophical grounds, for I do not defend here my ideas on these grounds. I merely suggest for the attention of teachers of mathematics devices to simplify their work. When we teach by drill that $- \times - = +$ we are using a similar method. It is not understanding that we aim to achieve, but a perceptional and active habit of replacing an even number of minus signs by a " + ".

Let us now consider the identity $A^2 - B^2 = (A-B)(A+B)$. The difficulty experienced by pupils lies in the recognition of the generality of the formula and of how much can be done with it. The method I use is first to get from

the pupils the fact that A and B are not essentials, since they can be replaced by X and Y or by any other letters we care to choose. What then remains is the pattern, $\ ^2 - \ ^2 = (\ -\ )(\ +\ )$, which can also be reversed,

$$(\quad - \quad)(\quad + \quad) = \quad ^2 - \quad ^2.$$

We then see what is identical on both sides. It becomes a recognition of perceptions. All that is then required is to see that a perception can be as wide as we choose to make it and that there is no limit to the complications we can reach. This obviously does not cover the case $A^4 - B^4$ or $A^8 - B^8$, since here another kind of perception is involved, that of $A^4 = (A^2)^2$, which must be grasped before we can concentrate on the second step of the process.

Let us pass on to quadratics: $ax^2 + bx + c = 0$. Literal quadratics and formulae for solutions are sometimes taught to third forms, and it is work at such a level that I have in mind here. We have all experienced the kind of difficulty met by pupils; for instance, the presence of x on both sides of the solution. Without going into details it may be said that it is usual to make pupils thoroughly familiar with the equation before attempting to solve the literal equation. I differ from most teachers in that I do not begin by solving numerical examples. My approach is to separate the visual impressions one gets from the equation in which all the letters seem of equal value. I write a, b, c, x, in different colours and also the exponent ^{2}, using say red yellow, green, white, blue, and white again for the signs $+$, $-$, $=$, 0. I give to a, b, c, different names, calling a Sheila's number, for instance, b Robert's, c John's. With their help I build up sets of equations and show that the arbitrariness of the coefficients is that of given numbers chosen at random. x remains untouched and is shown to be the outcome of the equation. This first acquaintance separates two planes, that of a, b, c, from that of x, then three planes, that of a from that of b and that of c. When we have agreed that it is more economical to work the literal equation than the ∞^3 of equations with numerical coefficients, we begin to see that there are various types of equations. In order to have quadratics a *must* differ from 0, but b and c can be 0. Also a, b, or c can be the same in various equations. We then write down the four types in colour:

$$\begin{aligned}
\text{I.}\quad & ax^2 + bx + c = 0,\\
\text{II.}\quad & ax^2 + bx \phantom{{}+c} = 0,\\
\text{III.}\quad & ax^2 \phantom{{}+bx} + c = 0,\\
\text{IV.}\quad & ax^2 \phantom{{}+bx+c} = 0,
\end{aligned}$$

and we see that *all* quadratic equations must fall into one of these classes. This full investigation of the meaning of the various terms of the equation frees the pupils' minds from all doubt as to what they are looking at. It seems to me essential, and I spend one or two periods with the class on this work which is usually considered to be unnecessary. It is only because my psychology of mathematical learning uses sensorial, active or emotional approaches to each question that I find it necessary to give such foundations to my teaching. So far I have found it profitable in that it saves much unnecessary repetition.

For the solution of the literal equations I begin by solving $ax^2 = 0$. As $a \neq 0$, the only possible answer is $x = 0$. This case acquaints the pupils with what we call solution of equations, finding the value of x, which by going through various operations indicated in the equation must reduce the whole to nothing. $ax^2 = 0$ requires but little thought, and may not show that the pupils have really understood what we are aiming at. But $ax^2 + c = 0$ is a much more profitable test of understanding. As only one operation is visible, the addition shown by $+$, we can direct the pupils' thoughts towards the idea that the

addition of two similar quantities can never produce nothing. The lesson is two-fold: (*a*) a quadratic equation may not always have a solution, and (*b*) ax^2 and c must be of opposite signs, or $ac<0$, from which we get that $-c/a>0$. This step is rather difficult to obtain from the pupils, since the minus sign is overwhelming and makes $ac<0$ count almost for nothing. But after some skilled questioning and direction we can familiarise them with $x^2=-c/a$ and with the simultaneous fact that $ac<0$. The persistence of the sign $-$ is really troublesome, and the trouble is extreme when we come to deal with $x=\pm\sqrt{(-c/a)}$, with its new element $\pm$. Most careful preparation of the ground is necessary here, to make clear its relation to what has already been mastered. This situation shows, by the way, how little of the algebra so far learnt has really been transcended.

Another type of difficulty is met in $ax^2+bx=0$, for although it is easy to factorise and $x(ax+b)=0$ has been dealt with as a product to be equated to 0, the pupils still think of x as *one* thing and we want them to discover that x could be either 0 or $-b/a$, which to them appear to be simultaneous incompatible values of one unknown. Our task consists in letting them see that what we are seeking is *a possible value for x and not to fix x in two ways.* In my experience of literal quadratics these difficulties are great, for they represent the understanding of what is abstract in all this. It should be noted that the solution of quadratics by factors is purely mechanical work and acceptable as a substitute only in the case of easy factors. The solution of (I) contains new difficulties, which can easily be described by the reduction of (I) to (III).

Quadratics have been dealt with in so much detail in order to enable my readers to have a clearer view of the way in which I undertake the analysis of the teaching of mathematics, seeking every possible device to ensure a thorough acquaintance with all that is implied in a formula, but never forgetting the elements in the child's mind which must be the bond between his experiences and what we offer for his attention.

In geometry the same process is followed, but space will not allow me to go further. In trigonometry at a higher level I have obtained results of the same kind, since in fact trigonometry is an algebra of a special kind using definite relations between symbols.

C. G.

1601. Why worry about Method?

Solve $(5-3x)(7-2x)=(11-6x)(3-x)$.

Answer:

$$5-3x+7-2x=11-6x+3-x,$$
$$12-5x=14-7x,$$
$$2x=2,$$
$$x=1.$$

[From a script.]

1602. Science is wonderful.

It must amuse a mathematician
Taking up his position
On the sort of weighing-machine
We've all seen
To reflect that its internal stress
Is $t=\pi\sqrt{\dfrac{2(P+Q)(l^2+a^2)+Gy^2}{g\{2(P+Q)a+Gs\}}}$.

J.B.B.—*Punch*, August 18, 1948.

NOTES ON OPERATORS $F=\sum\limits_{k=0}^{n} F_k(x)\left(\dfrac{d}{dx}\right)^k$.

BY P. T. LANDSBERG.

LET x be a real continuously variable quantity, and the functions $F_t (t=1, 2, \ldots)$ be arbitrary functions of x which are assumed to have derivatives of sufficiently high order in a given interval of x. Let $P\equiv d/dx$ so that P is skew-hermitian and satisfies

$$PF^{(y)}-F^{(y)}P=F^{(y+1)}, \qquad \text{.............................(1)}$$

where y is a non-negative integer and F is any F_t. We consider properties of operators of the type

$$A=\sum_0^n F_t P^t, \quad F_n\neq 0, \quad P^0=I, \qquad \text{..........................(2)}$$

where I is the identity operator.

Lemma 1. For any positive integer t,

$$P^t F=\sum_{s=0}^{t}\binom{t}{s}F^{(s)}P^{t-s}.$$

Proof. By (1) the lemma holds for $t=1$; we assume that it holds for $t=n$, and then

$$\begin{aligned}
P^{n+1}F &= \sum_{s=0}^{n}\binom{n}{s}PF^{(s)}P^{n-s}\\
&= \sum_{s=0}^{n}\binom{n}{s}\{F^{(s)}P+F^{(s+1)}\}P^{n-s}, \quad \text{by (1)}\\
&= \sum_{s=0}^{n}\binom{n}{s}F^{(s)}P^{n+1-s}+\sum_{r=1}^{n+1}\binom{n}{r-1}F^{(r)}P^{n+1-r}\\
&= FP^{n+1}+\sum_{u=1}^{n}\binom{n+1}{u}F^{(u)}P^{n+1-u}+F^{(n+1)}\\
&= \sum_{s=0}^{n+1}\binom{n+1}{s}F^{(s)}P^{n+1-s},
\end{aligned}$$

and by induction the lemma is proved.

The following elementary identity is also required:

Lemma 2. $x\equiv\sum\limits_{r=0}^{n}\sum\limits_{s=0}^{r}a(r,s)=\sum\limits_{r-s=0}^{n}\sum\limits_{s=0}^{n-(r-s)}a(r,s)=\sum\limits_{r-s=0}^{n}\sum\limits_{r=r-s}^{n}a(r,s).$

Proof. The definition of x shows that $r\geqslant s$, $r=s+u$, say. Thus if u is constant at u_0, $s=r-u_0$ can range from 0 ($r=u_0$) to $n-u_0$ ($r=n$). The possible values of u are 0, 1, 2, ... n. Hence the first equation of the lemma is obtained, the variable r having been replaced by the variable $u=r-s$. Similarly the variable s may be replaced by the variable $r-s$, as in the second equation of the lemma, since for $u=u_0$ the variable $r=s+u$ can vary from u_0 to n.

Turning to the operator (2), its hermitian conjugate is given by

$$\begin{aligned}
A^* &= \sum_0^n(-)^t P^t F_t^*\\
&= \sum_{t=0}^{n}\sum_{s=0}^{t}(-)^t\binom{t}{s}F_t^{*(s)}P^{t-s}, \quad \text{(lemma 1)}\\
&= \sum_{u=0}^{n}\sum_{t=u}^{n}(-)^t\binom{t}{u}F_t^{*(t-u)}P^{u}, \quad \text{(lemma 2)} \qquad \text{.............(3)}
\end{aligned}$$

where the asterisk denotes the conjugate complex number. The necessary and sufficient condition that A be a hermitian operator is, by (2) and (3),

$$F_u = \sum_{t=u}^{n} (-)^t \binom{t}{u} F^{*(t-u)}, \quad (u=0, 1, \ldots n). \qquad \ldots\ldots(4)$$

Hence we have a simple proof of a theorem recently given by H. C. Lee.†

The method may be applied to the determination of the conditions that two operators of type (2) commute.

Consider the arbitrary operators

$$F = \sum_0^n F_k P^k, \quad G = \sum_0^n G_k P^k,$$

with at least one of F_n, G_n not zero.

$$\begin{aligned} FG &= \sum_{t=0}^{n} \sum_{r=0}^{n} F_r P^r G_t P^t \\ &= \sum_{t=0}^{n} \sum_{r=0}^{n} \sum_{s=0}^{r} \binom{r}{s} F_r G_t^{(s)} P^{r+t-s}, \quad \text{by lemma 1}, \\ &= \sum_{t=0}^{n} \sum_{u=0}^{n} \sum_{r=u}^{n} \binom{r}{u} F_r G_t^{(r-u)} P^{t+u}, \end{aligned}$$

putting $r-s=u$ and using lemma 2. Thus

$$FG = \left[\sum_{v=0}^{n} \sum_{u=0}^{v} + \sum_{v=n+1}^{2n} \sum_{u=v-n}^{n}\right] \sum_{r=u}^{n} \binom{r}{u} F_r G_{v-u}^{(r-u)} P^v. \qquad \ldots\ldots(5)$$

where v has been put for $t+u$, and similarly, for GF. The condition for F and G to commute is therefore

$$\sum_{u=0}^{v} \sum_{r=u}^{n} \binom{r}{u} F_r G_{v-u}^{(r-u)} = \sum_{a=0}^{v} \sum_{b=a}^{n} \binom{b}{a} G_b F_{v-a}^{(b-a)}, \quad (v=0, 1, \ldots n), \quad \ldots\ldots(6)$$

$$\sum_{u=v-n}^{n} \sum_{r=u}^{n} \binom{r}{u} F_r G_{v-u}^{(r-u)} = \sum_{a=v-n}^{n} \sum_{b=a}^{n} \binom{b}{a} G_b F_{v-a}^{(b-a)}, \quad (v=n, n+1, \ldots 2n). \quad \ldots\ldots(7)$$

Hence

for $v=0$, $\quad \sum_{r=0}^{n} F_r G_0^{(r)} = \sum_{b=0}^{n} G_b F_0^{(b)}$.

or $\quad \sum_{r=1}^{n} \{F_r G_0^{(r)} - F_0^{(r)} G_r\} = 0$; $\quad \ldots\ldots(8a)$

for $v=1$, $\quad \sum_{r=0}^{n} \{F_r G_1^{(r)} - F_1^{(r)} G_r\} + \sum_{r=1}^{n} r\{F_r G_0^{(r-1)} - F_0^{(r-1)} G_r\} = 0$; $\quad \ldots\ldots(8b)$

....

for $v=2n-2$,

$$\sum_{r=n-2}^{n} \binom{r}{n-2} \{F_r G_n^{(r-n+2)} - F_n^{(r-n+2)} G_r\} + \sum_{r=n-1}^{n} \binom{r}{n-1} \{F_r G_{n-1}^{(r-n+1)} - F_{n-1}^{(r-n+1)} G_r\}$$
$$+ F_n G_{n-2} - F_{n-2} G_n = 0; \quad \ldots\ldots(8c)$$

for $v=2n-1$,

$$\sum_{r=n-1}^{n} \binom{r}{n-1} \{F_r G_n^{r-n+1} - F_n^{(r-n+1)} G_r\} + F_n G_{n-1} - F_{n-1} G_n = 0; \quad \ldots\ldots(8d)$$

for $v=2n$, $\quad F_n G_n - F_n G_n = 0$. $\quad \ldots\ldots(8e)$

† *Chinese Journ. Phys.*, 6, 86–99 (1946).

Condition (8*e*) is an identity and may be neglected ; (8*d*) yields

$$F_{n-1}G_n - F_nG_{n-1} + nF_nG_n' - nF_n'G_n + F_nG_{n-1} - F_{n-1}G_n = 0$$

or

$$F_n'/F_n = G_n'/G_n, \qquad \text{.....(9)}$$

Similarly (8*c*) yields

$$F_{n-1} = aG_{n-1} + bG_n^{1-1/n}. \qquad \text{.....(10)}$$

For $n=1$ (8*a*) which yields $F_1G_0' = F_0'G_1$, and (9) must be satisfied. This gives in the most general case $F_1 = aG_1$, $F_0 = aG_0 + b$, *i.e.* $F = aG + b$, where a and b are constants. For $n=2$ we have similarly by (9) and (10),

$$F_2 = aG_2, \quad F_1 = aG_1 + bG_2^{1/2},$$

and F_0 must satisfy (8*a*) and (8*b*); (8*a*) yields

$$F_0'G_1 + F_0''G_2 = G_0'(aG_1 + bG_2^{1/2}) + aG_0''G_2.$$

For larger values of n the explicit equations for the coefficients of F become rather complicated.

Lastly, the question of idempotency may be considered. The condition $F^2 = F$ for idempotency of an operator F entails with $F = G = \sum_1^n F_kP^k$, $(F_n \neq 0)$,

$$\sum_{u=v-n}^{n} \sum_{r=u}^{n} \binom{r}{u} F_rF_{v-u}^{(r-u)} = F_v, \quad (v = n, n+1, \ldots 2n),$$

where equal powers of P have been equated. For $v = 2n$, $u = r = n$ and

$$F_n^2 = F_{2n} = \begin{cases} 0, & n \geqslant 1 \\ F_0, & n = 0, \end{cases}$$

since for all cases for which $2n > n$, $F_{2n} = 0$. It follows that the algebra of the linear sets ΣF_rP^r contains no idempotent element except for the trivial ones $(F_0I)^2 = F_0I$, which are idempotent in the field of the coefficients. This circumstance is closely connected with the fact that different powers of P have been assumed to be linearly independent, so that the algebra of the linear sets ΣF_rP^r cannot have a finite basis. P. T. L.

1603. Spelling, even in his native German, was always shaky, and punctuation more shaky still: the mature Beethoven seldom ventured on anything more decisive than a comma. Arithmetic was beyond him. He might manage a little simple addition with the help of his ten fingers, but the calculations involved in financial transactions always gave him difficulty, and right at the end of his life the composer of the Ninth Symphony was being instructed by his nephew in simple multiplication—on his death-bed!—Peter Latham, "Beethoven", *Lives of the Great Composers*, ed. A. L. Bacharach. [Per Mr. R. H. Macmillan.]

1604. Peterlee will be a new town complete with every form of building that our civilization requires. . . . It will be a four-dimensional town, for not only must its surface pattern and heights be considered, but also the underground mine workings and coal.—*News-Chronicle*, May 10, 1948. [Per Mr. A. R. Pargeter.]

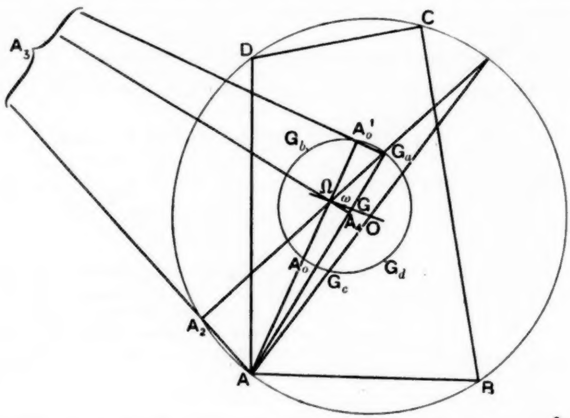
1605. It is necessary to appreciate that there are two distinct types of gas turbine at present in use for aircraft purposes, and of possible theoretical value on road vehicles. The first is that loosely described as the jet engine, in which the propulsive effort is obtained by the reactions on the atmosphere of the expelled gases after combustion.—J. Easton Gibson, "Gas Turbines for Motor Cars?" *Country Life*, May 14, 1948. [Per Mr. H. Martyn Cundy.]

SUR LE QUADRANGLE INSCRIT À UN CERCLE.

PAR VICTOR THÉBAULT.

APRÈS avoir signalé un nouveau cercle associé à un quadrangle plan quelconque nous nous proposons d'étendre au quadrangle inscrit à un cercle des propriétés du triangle et du tétraèdre signalées par G. de Longchamps, J. Griffiths et par nous-même.

Notations. Etant donné un quadrangle convexe $ABCD$ inscrit à un cercle (O, R) de rayon R, soient $BC=a$, $CA=b$, $AB=c$, $DA=a'$, $DB=b'$, $DC=c'$ les longuers de ses côtés et de ses diagonales. Nous appelons *médianes* du quadrangle les droites AG_a, BG_b, CG_c, DG_d joignant les sommets A, B, C, D aux centres de gravité G_a, G_b, G_c, G_d des triangles BCD, CDA, DAB, ABC. Le symétrique Ω du centre O du cercle (O, R) par rapport au centre G des moyennes distances des sommets A, B, C, D est l'*anticentre* du quadrangle.*



1. Considérons, d'abord, le cas d'un quadrangle plan quelconque; les distances du point A aux sommets et au barycentre G_a du triangle BCD vérifiant la relation bien connue

$$AB^2+AC^2+AD^2=BG_a^2+CG_a^2+DG_a^2+3AG_a^2,$$

on a

$$AG_a^2=\tfrac{1}{3}(b^2+c^2+a'^2)-\tfrac{1}{9}(a^2+b'^2+c'^2) \quad \text{.......................(i)}$$

et des formules déduites de celle-ci par des permutations circulaires sur a, b, c, a', b', c' donnent les carrés des autres médianes du quadrangle.

Rappelons aussi la relation †

$$\rho_a^2=\tfrac{1}{18}(a^2+b'^2+c'^2) \quad \text{..............................(ii)}$$

et celles que l'on obtient par des permutations circulaires sur a, b, c et a', b', c' qui expriment les carrés des rayons des cercles orthoptiques des ellipses de Steiner inscrites aux triangles BCD, CDA, DAB, ABC.

* Mathot, *Mathesis* (1901), 25. Ce point possède de nombreuses propriétés dont la bibliographie rappelle les noms de E. Lemoine, *Nouvelles Annales* (1869), 174 et 317, Deteuf, *id.* (1908), 442, ...

† J. Neuberg, *Association Française pour l'Avancement des Sciences* (1889), 176.

Théorème I. *Dans un quadrangle plan quelconque ABCD les cercles décrits sur les médianes comme diamètres rencontrent les cercles orthoptiques des ellipses de Steiner inscrites aux triangles BCD, CDA, DAB, ABC en huit points d'un même cercle.* (Cercle des huit points.)

En effet, M_a étant un des points de rencontre du cercle (G_a, ρ_a) avec le cercle (G_a') décrit sur la médiane AG_a comme diamètre, en observant que le point G partage la médiane AG_a dans le rapport $GA : GG_a = -3 : 1$ et bissecte la distance G_aG_a', on obtient

$$GM_a^2 = \tfrac{1}{2}(M_aG_a^2 + M_aG_a'^2 - \tfrac{1}{2}G_aG_a'^2),$$

d'où, après réductions et par analogie,

$$GM_a^2 = \tfrac{1}{48}\Sigma(a^2 + a'^2) = GM_b^2 = GM_c^2 = GM_d^2 = \sigma^2. \quad \text{..............(iii)}$$

Les huit points de rencontre des cercles (G_i, ρ_i), $(i = a, b, c, d)$ sont donc sur un même cercle (G, σ) (reél ou imaginaire) centré au barycentre G du quadrangle $ABCD$.

2. **Théorème II.** *Dans un quadrangle ABCD inscrit à un cercle (O, R), les centres de gravité des triangles BCD, CDA, DAB, ABC, les points* A_0, B_0, C_0, D_0 *situés aux tiers des distances de l'anticentre* Ω *aux sommets A, B, C, D et les points* A_0', B_0', C_0', D_0' *situés sur le prolongement de ces droites aux tiers des distances de* Ω *au cercle circonscrit sont douze points d'un même cercle* $(\omega, \frac{1}{3}R)$ *de rayon* $\frac{1}{3}R$ *centré au tiers de la distance* $G\Omega$. (Cercle des douze points.)

En effet, le quadrangle $G_aG_bG_cG_d$ étant le transformé du quadrangle $ABCD$ par l'homothétie $(G, -\frac{1}{3})$, qui entraine l'homothétie $(\Omega, \frac{1}{3})$, le cercle $(\omega, \frac{1}{3}R)$ correspond au cercle circonscrit (O, R) dans cette transformation.

Le théorème de Menelaus appliqué au triangle $A\Omega G_a$ et à la transversale $G_a\omega$ indique que les droites A_0G, B_0G, C_0G, D_0G sont des diamètres du cercle des douze points et que les points A_0', B_0', C_0', D_0' coïncident avec les projections orthogonales de G_a, G_b, G_c, G_d sur $A\Omega, B\Omega, C\Omega, D\Omega$.

D'autre part, on sait que $4R^2 = 4OA^2 = \Sigma GA^2 + 4OG^2$, d'où en vertu de (i),

$$OG^2 = G\Omega^2 = R^2 - \tfrac{1}{16}\Sigma(a^2 + a'^2). \quad \text{.........................(iv)}$$

Le plan radical (π) des cercles (O, R) et $(\omega, \frac{1}{3}R)$ rencontre OG en un point K tel que

$$GO^2 - R^2 - (G\omega^2 - \tfrac{1}{9}R^2) = 2 \,.\, O\omega \,.\, GK,$$

d'où

$$GK = \{\Sigma(a^2 + a'^2)\}/48OG. \quad \text{..............................(v)}$$

3. **Théorème III.** *Dans un quadrangle inscriptible ABCD les cercles décrits sur les médianes comme diamètres, les cercles orthoptiques des ellipses de Steiner inscrites aux triangles BCD, CDA, DAB, ABC et le cercle des huit points* (G, σ), *sont orthogonaux à un même cercle* (Ω, ρ) *de centre* Ω. (Cercle de Longchamps du quadrangle $G_aG_bG_cG_d$.)

En appliquant le théorème du carré de la médiane aux triangles $A\Omega O$, $B\Omega O$, $C\Omega O$, $D\Omega O$, puis le théorème de Stewart aux triangles $A\Omega G$, $B\Omega G$, $C\Omega G$, $D\Omega G$, on trouve

$$\Omega A^2 = R^2 + \tfrac{1}{4}(b^2 + c^2 + a'^2 - a^2 - b'^2 - c'^2), \quad \text{.....................(vi)}$$

et

$$\Omega G_a'^2 = R^2 - \tfrac{1}{9}(a^2 + b'^2 + c'^2) = OG_a^2, \quad \text{.....................(vii)}$$

puis

$$\Omega G_a^2 = R^2 - \tfrac{1}{36}\{3(b^2 + c^2 + a'^2) + a^2 + b'^2 + c'^2\}, \quad \text{...............(viii)}$$

Dès lors, les puissances

$$|\,\Omega G_a'^2 - \tfrac{1}{4}AG_a^2\,|, \ldots, |\,\Omega G_a^2 - \rho_a^2\,|, \ldots, |\,\Omega G^2 - \sigma^2\,|,$$

de l'anticentre Ω par rapport aux cercles en cause ont pour valeur commune

$$R^2 - \tfrac{1}{12}\Sigma(a^2 + a'^2) = \tfrac{1}{3} \mid MO^2 - R^2 \mid = \rho^2. \qquad \text{(ix)}$$

Le cercle (Ω, ρ) orthogonal à ces neuf cercles pourrait être appelé *cercle de Longchamps* du quadrangle inscriptible $G_aG_bG_cG_d$, par analogie avec le cercle * et la sphère † du même nom d'un triangle et d'un tétraèdre.

4. **Théorème IV.** *Dans un quadrangle inscriptible $ABCD$ les cercles orthoptiques des ellipses de Steiner inscrites aux triangles BCD, CDA, DAB, ABC sont orthogonaux aux cercles décrits sur les distances $A\Omega$, $B\Omega$, $C\Omega$, $D\Omega$ des sommets à l'anticentre comme diamètres.*

La puissance du centre G_a du cercle (G_a, ρ_a) par rapport au cercle (A_1) de centre A_1, décrit sur $A\Omega$ comme diamètre ayant pour expression

$$(P_a) = \mid G_aA_1^2 - \tfrac{1}{4}A\Omega^2 \mid = \tfrac{1}{2} \mid G_a\Omega^2 + AG_a^2 - A\Omega^2 \mid,$$

on a, en vertu des relations (i), (v), (vii) et après réductions,

$$(P_a) = \tfrac{1}{18}(a^2 + b'^2 + c'^2) = \rho_a^2, \dots,$$

et le théorème est démontré.

5. **Théorème V.** *Dans un quadrangle inscriptible $ABCD$ les cercles orthoptiques des cercles de Monge des ellipses de Steiner inscrites aux triangles BCD, CDA, DAB, ABC rencontrent le cercle circonscrit (O, R) en des points diamétralement opposés sur ces cercles.*

En effet, il résulte des formules (ii) et (vii) que

$$R^2 - OG_a^2 = \tfrac{1}{9}(a^2 + b'^2 + c'^2) = 2\rho_a^2, \dots .$$

6. Convenons d'appeler *cercle orthocentroïdal* du quadrangle inscriptible $ABCD$ le cercle (Γ) décrit sur $G\Omega$ comme diamètre, par analogie avec le cercle et la sphère de même nom d'un triangle (Tucker) et d'un tétraèdre.‡

Théorème VI. *Dans un quadrangle inscriptible $ABCD$, le double de la somme des carrés des côtés et de la diagonale aboutissant à un sommet diminué de la somme des carrés des côtés du triangle opposé est égal à huit fois la puissance de ce sommet par rapport au cercle orthocentroïdal (Γ).*

Les puissances des sommets A, B, C, D pour le cercle (Γ) sont égales à

$$\mid A\Gamma^2 - \tfrac{1}{4}G\Omega^2 \mid, \quad \mid B\Gamma^2 - \tfrac{1}{4}G\Omega^2 \mid, \quad \mid C\Gamma^2 - \tfrac{1}{4}G\Omega^2 \mid, \quad \mid D\Gamma^2 - \tfrac{1}{4}G\Omega^2 \mid.$$

En appliquant le théorème du carré de la médiane aux triangles $AG\Omega$, $BG\Omega$, $CG\Omega$, $DG\Omega$, on obtient, en raison des relations (i) à (viii)

$$\mid A\Gamma^2 - \tfrac{1}{4}G\Omega^2 \mid = \tfrac{1}{8}\{2(b^2 + c^2 + a'^2) - a^2 - b'^2 - c'^2\},$$

et des permutations circulaires sur a, b, c et a', b', c' donnent les puissances des autres sommets B, C, D par rapport au cercle (Γ).

Quadrangle inscriptible particulier. Pour que l'anticentre Ω d'un quadrangle inscriptible $ABCD$ soit situé sur le cercle circonscrit (O, R), il faut et il suffit que l'on ait, en vertu de la formule (iv),

$$\Sigma(a^2 + a'^2) = 12R^2.$$

Pour construire ce quadrangle étant donné le cercle circonscrit (O, R) et une position arbitraire du point Ω sur celui-ci, joignons un autre point A également arbitraire sur (O, R) au milieu G de $O\Omega$ et prolongeons la distance AG du tiers de sa longueur. Le point G_a ainsi obtenu coïncide avec le centre

* G. de Longchamps, *Journal de Mathématiques spéciales* (1886), 57.

† V. Thébault, *Mathesis* (1932), 223.

‡ V. Thébault, *Mathesis*, t. IV, 264.

de gravité d'une infinité de triangles BCD inscrits à (O, R) dont la construction classique est toujours possible car le point G_a décrit la circonférence de rayon $R/3$ centrée au tiers de ΩG à partir de Ω tangente à (O, R) au point donné Ω. Ce quadrangle spécial $Q \equiv ABCD$ dont le centre de l'hyperbole équilatère circonscrite est sur le cercle circonscrit possède de nombreuses propriétés dont nous retiendrons la suivante, par analogie avec un théorème relatif au tétraèdre que nous avons récemment démontré.*

Théorème VII. *Dans un quadrangle inscriptible $Q \equiv ABCD$ dont l'anticentre Ω est sur le cercle circonscrit, les cercles orthocentroïdaux des quadrangles $ABCD$, ΩBCD, ΩCDA, ΩDAB, ΩABC sont concourants.*

En effet, d'après le théorème précédent, la puissance du sommet Ω de l'anticentre du quadrangle $ABCD$ pour le cercle orthocentroïdal (Γ_a) du quadrangle inscrit ΩBCD a pour expression

$$\tfrac{1}{8}\{2(\Omega B^2 + \Omega C^2 + \Omega D^2) - BC^2 - CD^2 - DB^2\},$$

c'est-à-dire, en y introduisant les valeurs précitées de $\Omega B^2, \dots$,

$$\tfrac{1}{16}\{12R^2 - \Sigma(a^2 + a'^2)\} = 0.$$

Les cercles orthocentroïdaux (Γ_i) des quadrangles ΩBCD, ΩCDA, ΩDAB, ΩABC concourent donc en l'anticentre Ω du quadrangle $ABCD$ par où passe le cercle orthocentroïdal de celui-ci.

7. **Faisceau remarquable de cercles.** Les cercles (G_a'), (A_1) décrits sur AG_a, $A\Omega$ comme diamètres concourent en A et au point A_2 tel que la droite $A_2\Omega$ recoupe le cercle (O, R) au point diamétralement opposé au sommet A. Le cercle orthoptique (G_a, ρ_a) de l'ellipse de Steiner inscrite au triangle BCD étant orthogonal au cercle (A_1) décrit sur $A\Omega$ comme diamètre et au cercle de Longchamps (Ω, ρ) du quadrangle $G_aG_bG_cG_d$, l'axe radical de ces cercles, perpendiculaire à $A_1\Omega$, passe par G_a et rencontre AA_2 au centre radical A_3 des cercles (G_a'), (A_1), (Ω, ρ), (O, R). L'anticentre Ω se confondant visiblement avec l'orthocentre du triangle $A\Omega A_3$, les droites AG et $A_3\Omega$ sont perpendiculaires en un point A_4 du cercle orthocentroïdal (Γ) du quadrangle $ABCD$. Dès lors

$$A_3A_2 \,.\, A_3A = A_3\Omega \,.\, A_3A_4$$

et les cercles (O, R), (Γ), (Ω, ρ) dont les centres O, Γ, Ω sont collinéaires appartiennent déjà à un faisceau (F) dont l'axe perpendiculaire à la base $O\Omega$ passe par A_3 et les points analogues B_3, C_3, D_3.

De plus, l'anticentre Ω ayant même puissance pour les cercles (G_a'), (G_a, ρ_a) et (G, σ), l'axe radical des deux premiers cercles passe par G et le cercle des huit points (G, σ) est un quatrième cercle du faisceau (F).

Si l'axe radical des cercles (G, σ) et (O, R) rencontre la droite OG en un point K', on obtient, d'après un théorème connu

$$-\sigma^2 - |\, GO^2 - R^2 \,| = 2GO \,.\, GK',$$

c'est-à-dire, en raison des relations (iii), (iv), (v),

$$GK' = \{\Sigma(a^2 + a'^2)\}/48OG = GK.$$

Théorème VIII. *Dans un quadrangle inscriptible $ABCD$, les extrémités des diamètres des cercles orthoptiques des ellipses de Steiner inscrites aux triangles BCD, CDA, DAB, ABC perpendiculaires à ΩG_a, ΩG_b, ΩG_c, ΩG_d sont sur un même cercle appartenant au faisceau (F).*

* V. Thébault, *Comptes-Rendus*, juin 1947.

Considérons, en effet, l'anticentre Ω_1 du quadrangle inscriptible $G_aG_bG_cG_d$ pour observer que

$$\Omega_1G_a = \tfrac{1}{3}A\Omega, \quad \Omega_1G_b = \tfrac{1}{3}B\Omega, \quad \Omega_1G_c = \tfrac{1}{3}C\Omega, \quad \Omega_1G_d = \tfrac{1}{3}D\Omega.$$

D'après les formules (ii) et (vi), il vient

$$\Omega_1{G_a}^2 + {\rho_a}^2 = \tfrac{1}{9}\{R^2 + \tfrac{1}{4}\Sigma(a^2 + a'^2)\} = \Omega_1{G_b}^2 + \rho^2{}_b = \ldots = \tau^2.$$

Or si l'axe radical des cercles (G, σ) et (Ω_1, τ) rencontre la droite OG en un point K'', on obtient

$$G{\Omega_1}^2 - \tau^2 + \sigma^2 = 2G\Omega_1 \,.\, GK'',$$

et

$$GK'' = \{\Sigma(a^2 + a'^2)\}/48OG = GK' = GK.$$

Le cercle (Ω_1, τ) fait donc aussi partie du faisceau (F). En résumé, on peut énoncer cette proposition :

Théorème IX. *Dans un quadrangle inscriptible $ABCD$, le cercle circonscrit (O, R), le cercle des douze points $(\omega, \tfrac{1}{3}R)$, le cercle orthocentroïdal (Γ), le cercle des huit points (G, σ), le cercle de Longchamps (Ω, ρ) du quadrangle $G_aG_bG_cG_d$ et le cercle (Ω_1, τ) appartiennent à un même faisceau.*

8. **Notes.** 1. J. Griffiths * et V. Thébault † ont signalé les faisceaux de cercles et de sphères analogues associés à un triangle et à un tétraèdre.

2. Mathot (*loc. cit.*) a montré que l'anticentre Ω du quadrangle $ABCD$ coïncide avec le point de concours des perpendiculaires menées par le milieu d'un côté ou d'une diagonale sur le côté ou la diagonale opposé.

Cette construction découle d'un cas particulier du théorème suivant que nous avons étendu au quadrangle gauche et au tétraèdre.‡

Théorème X. *Si les perpendiculaires menées par les milieux des côtés du quadrangle plan $ABCD$ sur les côtés correspondants (ou opposés) du quadrangle plan $A'B'C'D'$ concourent en un point Ω', les perpendiculaires menées des milieux des côtés du quadrangle $A'B'C'D'$ sur les côtés correspondants (ou opposés) du quadrangle $ABCD$ concourent en un point Ω.*

Les quadrangles $ABCD$ et $A'B'C'D'$ sont *orthologiques par les milieux des côtés correspondants (ou opposés)* ; les points Ω' et Ω sont leurs *centres d'orthologie* respectifs.

Quand les quadrangles $ABCD$ et $A'B'C'D'$ sont inscrits à un cercle (O, R) et coïncident entre eux, les centres d'orthologie Ω, Ω' par les milieux des côtés correspondants se confondent avec le centre O du cercle circonscrit et les centres d'orthologie par les milieux des côtés opposés coïncident avec le symétrique du centre O du cercle circonscrit par rapport au barycentre G des sommets des quadrangles $ABCD \equiv A'B'C'D'$.

On retrouve ainsi la détermination classique de l'anticentre d'un quadrangle inscrit à un cercle. V. T.

* *Nouvelles Annales* (1864), 345 et (1865), 522.

† *Comptes-Rendus du Congrès international*, Oslo (1936), 142, et *L'Enseignement Mathématique* (1937), 98.

‡ *Annales de la Société scientifique de Bruxelles* (1947), 113.

1606. Cobden's monomania.

His late speech at Manchester is like that of a learned and ingenious mathematician endeavouring to prove animal-magnetism on geometrical principles. There is such an air of cogency and sense about his arguments, that while we read we are almost betrayed into forgetting their absolute inapplicability to his one great conclusion.—*The Times*, " A Hundred Years Ago ", from *The Guardian* of Feb. 2, 1848. [Per Mr. J. T. Combridge.]

A CONDITION FOR RIGHT-ANGLED TRIANGLES.

By B. D. Price.

It is well known that the identity

$$(x^2+y^2)^2 \equiv (x^2-y^2)^2+(2xy)^2$$

enables one to generate right-angled triangles with integral sides merely by putting x and y as integers $(x>y)$. It is sufficient that x, y be integers, to give a right-angled triangle with hypotenuse x^2+y^2 and sides x^2-y^2, $2xy$. Is the condition necessary? No, for the triangle with sides 9, 12, 15 (a multiple of the 3, 4, 5 case) cannot be expressed by the above identity without using surds. We will, however, prove the following proposition :

"It is necessary and sufficient that a prime right-angled triangle (with integral sides) be produced by substituting integers for x and y in the above identity, with the following conditions :

(i) $x>y$, (ii) $x-y$ is odd, (iii) x and y have no common factor."

A "prime" right-angled triangle is one in which there is no common factor of the three sides.

Suppose two of the sides have a common factor a. Let them be ab and ac (all letters are taken to mean integers). Then if d is the third side,

$$d=a\sqrt{(\pm b^2 \pm c^2)},$$

and so d is divisible by a also. Hence if a triangle is prime, any two of its sides are prime to one another.

From considerations of oddness or evenness of squares, it is clear that the only possible cases are :

	Hypotenuse	*Side*	*Side*
Case 1	Even	Even	Even
Case 2	Even	Odd	Odd
Case 3	Odd	Odd	Even

Case 1 does not give a prime triangle. Case 2 may be proved impossible. For supposing the hypotenuse be $2p$ and the other sides $2q+1$, $2r+1$, then

$$(2p)^2=(2q+1)^2+(2r+1)^2,$$

which leads to the contradiction

$$4p^2=4(q^2+q+r^2+r)+2,$$

where one side is divisible by 4 and the other not.

Hence all prime triangles belong to case 3.

Thus any prime triangle may be constructed :

Hypotenuse	$2a+1$
Side	$2b+1$
Side	$2c$,

by choosing suitable integers a, b, c. The cases a or $b=0$ are obviously impossible.

Consider a prime triangle with the above sides. We have proved that any two sides of it must be prime to one another. We need to prove that $(a+b+1)$ and $(a-b)$ are prime to one another. Suppose they have a common factor $f(\neq 1)$. Let

$$a+b+1=fg,$$
$$a-b=fh.$$

Then adding, $\qquad 2a+1=f(g+h)$,

and subtracting $\qquad 2b+1=f(g-h)$;

hence two sides of the prime triangle have a common factor f, which is absurd. Hence $(a+b+1)$ and $(a-b)$ are prime to one another.

Now, by Pythagoras' Theorem,

$$\begin{aligned}(2c)^2&=(2a+1)^2-(2b+1)^2\\&=(2a+2b+2)(2a-2b),\\c^2&=(a+b+1)(a-b).\end{aligned}$$

Since the factors on the right-hand side are prime to one another they must both be perfect squares.

Now let $\qquad a+b+1=x^2,$
$\qquad a-b=y^2.$

Then
$$\begin{aligned}x^2-y^2&=2a+1,\\x^2-y^2&=2b+1,\\2xy&=2\surd\{(a+b+1)(a-b)\}\\&=2c;\end{aligned}$$

and so the numbers x and y as defined above generate the prime triangle. Hence any prime triangle may be generated by the identity.

x and y cannot be both even or both odd, as then the triangle obviously would not be prime. It remains to show that if x and y have no common factor (necessarily odd) they cannot generate a non-prime triangle.

Suppose they generate a non-prime triangle in which all sides are divisible by p.

Then let $\qquad x^2+y^2=pm,$
$\qquad x^2-y^2=pn.$

Therefore $\qquad 2x^2=p(m+n),$
$\qquad 2y^2=p(m-n).$

Thus both x and y are divisible by p or its square root. An unprime triangle cannot be generated if the conditions are observed.

All possible prime triangles may thus be generated by an "infinite table", of which I give the top left-hand corner. The first two numbers are x and y, the three below the sides of the triangle.

2, 1 3, 4, 5	3, 2 13, 12, 5	4, 3 25, 24, 7	5, 4 41, 40, 9
4, 1 17, 15, 8	5, 2 29, 21, 20	8, 3 73, 55, 48	7, 4 65, 56, 33
6, 1 37, 35, 12	7, 2 53, 45, 28	10, 3 109, 91, 60	9, 4 97, 72, 65
8, 1 65, 63, 16	9, 2 85, 77, 36	14, 3 205, 187, 84	11, 4 137, 105, 88

B. D. P.

1607. You may pay more at the market, but the directors of Covent Garden Opera House have just discovered that the price of their seats 100 years ago was the same, and in some cases more, than it is today in terms of hard cash. In terms of value a visit to the opera or ballet is more than 100 per cent. cheaper than it was in the prosperous day of 1848.—*News-Chronicle*, February 17, 1948. [Per Mr. F. Sandon.]

MATHEMATICAL NOTES.

2049. *On Notes* 1429 *and* 1457.

The previous Notes on the construction of rational functions of the form $(ax^2+2bx+c)/(a'x^2+2b'x+c')$ with given maximum and minimum values deal only with a few numerical cases and have missed a simple standard form $(ax+b)/(x^2+c)$ to which the general case can always be reduced by a change of origin. What essentially matters is not so much the actual values of the maximum and the minimum but their difference, which we may take to be *unity* by adjusting the scale for measuring the value of the function. We define $|\sqrt{(b^2+a^2c)}/c|$ as the *scale* for the function $(ax+b)/(x^2+c)$ and assume b^2+a^2c to be positive. If b^2+a^2c be negative, c must be negative and the function can assume all values, so that there is no question of a maximum or minimum.

We now proceed to investigate the function with rational coefficients given by

$$y=(ax+b)/(x^2+c). \qquad \text{.................................(1)}$$

We get at once the quadratic in x,

$$x^2y-ax+(cy-b)=0,$$

whose discriminant

$$a^2+4by-4cy^2, \qquad \text{.......................................(2)}$$

should be positive.

Case I : $c<0$. When $b^2+ca^2>0$, y does not take values between the limits

$$\{b\pm\sqrt{(b^2+a^2c)}\}/2c. \qquad \text{................................(3)}$$

These limits differ by $\sqrt{(b^2+a^2c)}/c$. We may now equate this difference to unity and thus fix the scale for the function and obtain the relation

$$b^2+a^2c-c^2=0. \qquad \text{......................................(4)}$$

Hence $$c=\tfrac{1}{2}\{a^2\pm\sqrt{(a^4+4b^2)}\},$$

and we choose the negative sign for the radical since $c<0$. Further, a^4+4b^2 should be a rational square and so

$$a^2=\lambda(m^2-n^2), \quad b=\lambda mn, \quad c=-\lambda n^2, \qquad \text{.....................(5)}$$

where m and n are integers prime to each other and λ is any rational number so chosen that $\lambda(m^2-n^2)$ is the square of a rational number.

Case II : $c>0$. In this case, y takes values between the limits

$$\{b\pm\sqrt{(b^2+a^2c)}\}/2c,$$

and we choose

$$c=\tfrac{1}{2}\{a^2+\sqrt{(a^4+4b^2)}\},$$

with a and b as before :

$$a^2=\lambda(m^2-n^2), \quad b=\lambda mn, \quad c=\lambda m^2. \qquad \text{........................(6)}$$

From (5) and (6) it is readily seen that we can make a and the ratio $b:c$ arbitrary, thereby fixing the values of m, n and λ thus :

I. When c is negative, we write

$$c=-a^2n^2/(m^2-n^2), \quad b=a^2mn/(m^2-n^2), \quad (m>n),$$

leaving a arbitrary and choosing least integers m and n such that $m:n=-b:c$. The required function is therefore

$$\left\{ax+\frac{a^2mn}{m^2-n^2}\right\}\Big/\left(x^2-\frac{a^2n^2}{m^2-n^2}\right), \qquad \text{.........................(7)}$$

which cannot take values between $-(m/2n)\pm\tfrac{1}{2}$.

II. When c is positive, we write

$$c=a^2m^2/(m^2-n^2), \quad b=a^2mn/(m^2-n^2), \quad (m>n),$$

leaving a arbitrary and choosing least integers m and n such that $m:n=c:b$. The required function for this case is

$$\left\{ax+\frac{a^2mn}{m^2-n^2}\right\}\bigg/\left(x^2+\frac{a^2m^2}{m^2-n^2}\right), \quad \text{.........................(8)}$$

with limits
$$\frac{n}{2m}\pm\frac{1}{2}.$$

From these standard functions (7) and (8) we easily construct a function with given limits α, β $(\alpha<\beta)$ by changing the origin and adjusting the scale thus:

$$y=(\beta-\alpha)\left\{ax+\frac{a^2mn}{m^2-n^2}\right\}\bigg/\left(x^2-\frac{a^2n^2}{m^2-n^2}\right)+\left(\frac{m}{2n}+\frac{1}{2}\right)(\beta-\alpha)+\alpha, \quad \text{...(9)}$$

which cannot take values between

$$(\beta-\alpha)\left(-\frac{m}{2n}-\frac{1}{2}\right)+\left(\frac{m}{2n}+\frac{1}{2}\right)(\beta-\alpha)+\alpha=\alpha,$$

$$(\beta-\alpha)\left(-\frac{m}{2n}+\frac{1}{2}\right)+\left(\frac{m}{2n}+\frac{1}{2}\right)(\beta-\alpha)+\alpha=\beta.$$

In (9) we may change x to $x-p$, where p is any rational number, without altering the limits of the function.

Similarly, the function which takes values only between α and β is

$$y=(\beta-\alpha)\left\{ax+\frac{a^2mn}{m^2-n^2}\right\}\bigg/\left(x^2+\frac{a^2m^2}{m^2-n^2}\right)+\left(\frac{1}{2}-\frac{n}{2m}\right)(\beta-\alpha)+\alpha. \quad \text{...(10)}$$

Putting $m=1$, $n=0$, $a=1$ in (10) with $\alpha=5$, $\beta=9$, we get, changing x to $x-\frac{1}{2}$,

$$y=(28x^2-12x+27)/(4x^2-4x+5),$$

which is the example given by Mr. Holmes.

Lastly, we show how we can readily write down the rational limits, if any, for a rational function of the form

$$(ax^2+bx+c)/(a'x^2+b'x+c').$$

Consider another example of Mr. Holmes, namely

$$y=(42x^2+24x-3)/(10x^2+8x+1).$$

Put $y=4\cdot2+Y'$ and $X=x+0\cdot4$. The function reduces to

$$Y'=(-9\cdot6X-3\cdot36)/(10X^2-0\cdot6).$$

The scale for the function is found to be 4. Take

$$Y=\tfrac{1}{4}Y'=(-2\cdot4X-0\cdot84)/(10X^2-0\cdot6),$$

which is the final reduced form for which the limits are

$$\frac{0\cdot84}{0\cdot6\times2}\pm\frac{1}{2},$$

that is, $0\cdot2$ and $1\cdot2$. Hence the limits for $y=4\cdot2+4Y$ are $4\cdot2+0\cdot8$ and $4\cdot2+4\cdot8$, that is, 5 and 9, as they ought to be.

Of course, the above method is meant only as a verification and not as a regular method to be taught to students for general adoption.

What we wish to observe is that any function of the form

$$(ax^2 + bx + c)/(a'x^2 + b'x + c)$$

can always be reduced to the form $(ax + b)/(x^2 + c)$ by a linear transformation involving only change of origin and change of y-scale and the limits for the latter function (differing by unity) will be rational only when $a^4 + 4b^2$ is a perfect square. In this case $b^2 + a^2c$ is also a square.

A. A. Krishnaswami Ayyangar.

2050. *The vector triple product.*

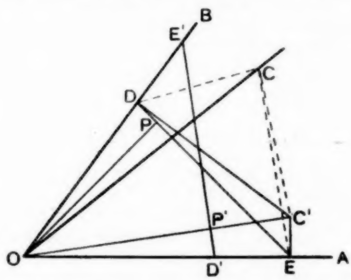
In this note we give a geometrical proof of the vector triple product formula

$$\mathbf{C}_\wedge(\mathbf{B}_\wedge\mathbf{A}) = (\mathbf{C}\,.\,\mathbf{A})\,\mathbf{B} - (\mathbf{C}\,.\,\mathbf{B})\,\mathbf{A},$$

where the sign $_\wedge$ denotes vector multiplication and the dot scalar multiplication.

Let **OA, OB, OC** be three vectors **A, B, C**; let θ be the angle AOB, and ϕ the angle between OC and the vector product $(\mathbf{B}_\wedge\mathbf{A})$.

Draw CC' perpendicular to the plane AOB and $C'D$, $C'E$ perpendicular to OB and OA respectively. Then it is easily seen that CD, CE are perpendicular



to OD, OE respectively. Set OD' $(=OD)$ along OA and OE' $(=OE)$ along OB. Draw OP, OP' perpendicular to DE and $D'E'$ respectively. From the congruence of the triangles ODE, $OD'E'$, we have $DE = D'E'$; OP and OP' make equal angles with OD and OE; OP being perpendicular to DE, OP' passes through the circumcentre of the triangle ODE. But OC' is the circum-diameter of ODE. Hence OP' falls on OC' and $D'E'$ is perpendicular to OC'.

Further, $$DE = OC' \sin\theta = OC \sin\theta \sin\phi, \quad\text{...........................(i)}$$

since $OC' = OC \sin\phi$.

Now, $$(\mathbf{C}\,.\,\mathbf{B}) = OC\,.\,OB \cos COB = OB\,.\,OD,$$

CD being perpendicular to OB.

Similarly, $$(\mathbf{C}\,.\,\mathbf{A}) = OA\,.\,OE.$$

Thus $$(\mathbf{C}\,.\,\mathbf{A})\,\mathbf{B} - (\mathbf{C}\,.\,\mathbf{B})\,\mathbf{A} = OA\,.\,OB(\mathbf{OE}' - \mathbf{OD}')$$

$$= OA\,.\,OB\,.\,\mathbf{D'E'}. \quad\text{........................(ii)}$$

It is well known that $\mathbf{C}_\wedge(\mathbf{B}_\wedge\mathbf{A})$ is a vector perpendicular to the plane containing OC and the normal at O to the plane AOB. It is, therefore, perpendicular to the plane OCC' and so to OC' or parallel to $D'E'$ in the plane AOB itself. The module of the vector is readily seen to be, by (i),

$$OA\,.\,OB\,.\,OC \sin\theta \sin\phi = OA\,.\,OB\,.\,DE = OA\,.\,OB\,.\,D'E', \quad\text{.........(iii)}$$

since $DE = D'E'$. From (ii) and (iii) we get

$$\mathbf{C}_\wedge(\mathbf{B}_\wedge\mathbf{A}) = (\mathbf{C} \,.\, \mathbf{A})\,\mathbf{B} - (\mathbf{C} \,.\, \mathbf{B})\,\mathbf{A}.$$

A. A. K. AYYANGAR.

2051. *A geometrical proof of the vector triple product formula.*

Various algebraic proofs of the vector triple product formula have been given in recent years, *e.g.* by Chapman and Milne, *Mathematical Gazette*, 23 (1939), 35. The object of the present Note is to present a concise proof of a more geometrical character. Incidentally, we also obtain at the same time the formula for the volume of a tetrahedron in terms of three concurrent edges and the angles between them.

From the geometrical definition of a vector product, it follows that $\mathbf{a}_\wedge(\mathbf{b}_\wedge\mathbf{c})$ must lie in the plane of $\mathbf{b}$ and $\mathbf{c}$; and, since its scalar product with $\mathbf{a}$ is zero, we immediately deduce that

$$\mathbf{a}_\wedge(\mathbf{b}_\wedge\mathbf{c}) = \lambda\{\mathbf{b}(\mathbf{c} \,.\, \mathbf{a}) - \mathbf{c}(\mathbf{a} \,.\, \mathbf{b})\}, \quad \text{.............................(1)}$$

where λ is a scalar.

To determine λ, it is clearly sufficient from now on to regard $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ as unit vectors. Let α be the angle between $\mathbf{b}$ and $\mathbf{c}$, β the angle between $\mathbf{c}$ and $\mathbf{a}$, γ the angle between $\mathbf{a}$ and $\mathbf{b}$, and θ the angle between $\mathbf{a}$ and $\mathbf{b}_\wedge\mathbf{c}$. Then, on squaring each side of (1), we find that

$$\sin^2\theta \sin^2\alpha = \lambda^2(\cos^2\beta + \cos^2\gamma - 2\cos\alpha\cos\beta\cos\gamma), \quad \text{..................(2)}$$

whence,

$$\cos^2\theta \sin^2\alpha = 1 - \cos^2\alpha - \lambda^2(\cos^2\beta + \cos^2\gamma - 2\cos\alpha\cos\beta\cos\gamma). \quad \text{.........(3)}$$

If V is the volume of the tetrahedron formed by the three unit vectors, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, the expression on the left-hand side of (3) represents $36V^2$. Hence, this must be the value of the expression on the right-hand side. But the formula for $36V^2$, in terms of α, β and γ, must be symmetrical. Hence λ^2 must be unity.

Finally, by drawing a figure, it is clear that λ must be positive, and hence it too must be unity.

G. J. WHITROW.

2052. *The vector triple product.*

The following proof of the formula

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \,.\, \mathbf{c})\,\mathbf{b} - (\mathbf{a} \,.\, \mathbf{b})\,\mathbf{c}$$

depends only on elementary plane geometry.

Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be represented by segments of lines **OA**, **OB**, **OC**, and let OP be the projection of OA on the plane BOC. Take $\angle BOC = \phi$, and take θ to be the angle between OA and the upward drawn normal to the plane BOC. Then $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is a vector of magnitude $abc \sin\phi \sin\theta$, and its direction is parallel to the plane BOC and perpendicular to OP.

Construction. Draw $OP' = OP$ such that $\angle BOP = \angle P'OC$ and draw PM, $P'N'$ perpendicular to OB, and PN, $P'M'$ perpendicular to OC.

It is to be proved that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = bc\mathbf{M'N'}.$$

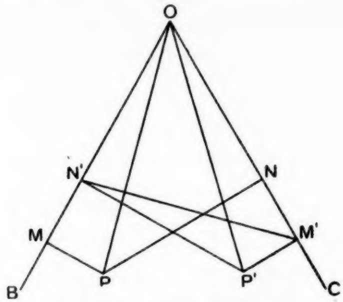
OP' is a diameter of the circle through $OM'P'N'$, so that

$$M'N' = O'P' \sin\phi = OP \sin\phi = a \sin\theta \sin\phi.$$

Also $$\angle P'N'M' = \angle P'OM' = \angle N'OP,$$

so that $M'N'$ is perpendicular to OP.

Further, with the diagram as drawn, θ and ϕ being supposed acute, $\mathbf{b} \times \mathbf{c}$ is along the upward normal to the plane, and $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ crosses OP from right to left, so that $bc\mathbf{M'N'}$ represents $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ completely.



But
$$bc\mathbf{M'N'} = bc\mathbf{ON'} + bc\mathbf{M'O},$$
and
$$cON' = cON = \mathbf{a} \,.\, \mathbf{c},$$
$$bOM' = bOM = \mathbf{a} \,.\, \mathbf{b},$$
so that
$$bc\mathbf{M'N'} = (\mathbf{a} \,.\, \mathbf{c})\,\mathbf{b} - (\mathbf{a} \,.\, \mathbf{b})\,\mathbf{c}.$$

R. J. LYONS.

2053. *Normals to a parabola.*

1. Referring to the point $(at^2, 2at)$ on the parabola $y^2 = 4ax$ as the point t, it is well known that the equation to the normal to the parabola at that point is

$$tx + y = 2at + at^3. \qquad \text{.......(1)}$$

In particular, if (x, y) is a fixed point three normals can in general be drawn from it to the parabola, and the roots t_1, t_2 and t_3 of (1) determine the co-ordinates of the feet of the normals.

2. If any three points t_1, t_2, t_3 are taken on the parabola the normals at these points are respectively

$$\left.\begin{aligned} t_1x + y &= 2at_1 + at_1^3, \\ t_2x + y &= 2at_2 + at_2^3, \\ t_3x + y &= 2at_3 + at_3^3. \end{aligned}\right\} \qquad \text{.......(2)}$$

These normals will be concurrent if, omitting the factor a,

$$\begin{vmatrix} t_1(2 + t_1^2) & t_1 & 1 \\ t_2(2 + t_2^2) & t_2 & 1 \\ t_3(2 + t_3^2) & t_3 & 1 \end{vmatrix} = 0,$$

which reduces to

$$\begin{vmatrix} t_1^3 & t_1 & 1 \\ t_2^3 & t_2 & 1 \\ t_3^3 & t_3 & 1 \end{vmatrix} = 0. \qquad \text{.......(3)}$$

3. The origin is at the vertex of the parabola, and any circle through the origin is of the form $x^2 + y^2 + 2gx + 2fy = 0$.

The points t_1, t_2, t_3 will lie on this circle if the equations

$$\left.\begin{aligned}(a^2t_1^4 + 4a^2t_1^2) + 2g \,.\, at_1^2 + 4f \,.\, at_1 = 0,\\ (a^2t_2^4 + 4a^2t_2^2) + 2g \,.\, at_2^2 + 4f \,.\, at_2 = 0,\\ (a^2t_3^4 + 4a^2t_3^2) + 2g \,.\, at_3^2 + 4f \,.\, at_3 = 0,\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

are satisfied.

Hence the condition that the points should lie on the circle is (omitting the common factors a^2 and a) :

$$\begin{vmatrix} t_1^4 + 4t_1^2 & t_1^2 & t_1 \\ t_2^4 + 4t_2^2 & t_2^2 & t_2 \\ t_3^4 + 4t_3^2 & t_3^2 & t_3 \end{vmatrix} = 0,$$

which reduces to

$$\begin{vmatrix} t_1^3 & t_1 & 1 \\ t_2^3 & t_2 & 1 \\ t_3^3 & t_3 & 1 \end{vmatrix} = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Hence, since (3) and (5) are identical conditions, it follows that, if the normals to a parabola at three points on it are concurrent, then their feet lie on a circle which passes through the vertex.

In view of the deduction from (1) above, this may be stated in the more usual form : Three normals can, in general, be drawn from a point to a parabola, and their feet lie on a circle which passes through the vertex of the parabola.

4. Factorising the determinant, the conditions (3) and (5) may be written

$$(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)(t_1 + t_2 + t_3) = 0,$$

whence, since t_1, t_2 and t_3 are different, we have

$$t_1 + t_2 + t_3 = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

We have now the result that if three normals to a parabola are concurrent (or if the three normals are drawn from a given point to a parabola), the sum of the ordinates of their feet is zero. The result follows also, of course, from equation (1) when (x, y) is a fixed point, since the equation contains no term in t^2.

5. Solving, say, the first and second equations of (2) we find

$$x = 2a + a(t_1^2 + t_1t_2 + t_2^2),$$
$$y = -at_1t_2(t_1 + t_2).$$

Now, for concurrent normals, by (6)

$$2(t_1^2 + t_1t_2 + t_2^2) = t_1^2 + t_2^2 + (t_1 + t_2)^2 = t_1^2 + t_2^2 + t_3^2 ; \quad \ldots\ldots\ldots\ldots(7)$$

and it follows that

$$\left.\begin{aligned} x &= 2a + \tfrac{1}{2}a\Sigma t_1^2,\\ y &= at_1t_2t_3. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

These values are symmetrical in t_1, t_2 and t_3, and determine the point of concurrence of the normals.

It follows from the expression for x in (8) that if three normals to a parabola are concurrent (or if the three normals are drawn from a fixed point to a parabola) the sum of the abscissae of their feet is $2(x - 2a)$, where x is the abscissa of the point of concurrence.

6. To find the equation to the circle through the feet of concurrent normals, let us solve, say, the first and second equations of (4) for g and f.

We find that
$$g = -2a - \tfrac{1}{2}a(t_1^2 + t_1t_2 + t_2^2)$$
$$= -2a - \tfrac{1}{4}a\Sigma t_1^2, \quad \text{by (7)},$$
and
$$f = \tfrac{1}{4}at_1t_2(t_1 + t_2)$$
$$= -\tfrac{1}{4}at_1t_2t_3, \quad \text{by (6)},$$
and these values are symmetrical in t_1, t_2, t_3.

The circle then is
$$x^2 + y^2 - 2x(2a + \tfrac{1}{4}a\Sigma t_1^2) - \tfrac{1}{2}ay\, t_1t_2t_3 = 0,$$
or
$$2(x^2 + y^2) - ax(8 + \Sigma t_1^2) - ay\, t_1t_2t_3 = 0. \qquad \text{(9)}$$

7. If t_1, t_2, t_3 are the feet of the normals drawn from a fixed point (α, β) to the parabola, then,

by (8),
$$2(\alpha - 2a) = a\Sigma t_1^2,$$
$$\beta = at_1t_2t_3;$$
and, from (9), the circle is
$$2(x^2 + y^2) - 2x(2a + \alpha) - \beta y = 0. \qquad \text{(10)}$$
This is the result given by Mr. N. M. Gibbins in Note 1862, *Gazette*, February, 1946.

The reader may also find some interest in the discussion in Charles Smith's *Conic Sections* (*Coordinate Geometry*), 1910 Edition, Section 106, leading to the equation (10) above.

8. There does not seem to be an easy elementary proof by the methods of pure geometry of the theorem that the feet of concurrent normals to the parabola lie on a circle through the vertex. The only " pure " proof known to the writer involves the methods of projective geometry and is the limiting case of Joachimsthal's Theorem on the normals to an ellipse : If four normals to an ellipse are concurrent the circle through the feet of three of them passes through the point diametrically opposite to the foot of the fourth. The theorem is also easy to prove analytically.

J. P. McCarthy.

2054. *Another approach to* π.

It is quite usual to define π as the smallest positive zero of the solution to the differential equation $y'' = -y$ which satisfies $y(0) = 0$, $y'(0) = 1$. We can expect to get an approximation to π (and to $\sin x$) by replacing the differential equation by a difference equation and solving it instead. If we take an interval of length 1 in x, then we may replace $y''(n)$ by
$$y(n+1) - 2y(n) + y(n-1),$$
and the differential equation by $y(n+1) = y(n) - y(n-1)$. The initial condition $y(0) = 0$ applies and $y'(0) = 1$ may be replaced by $y(1) = 1$. We thus obtain a simple recurrence relation for $y(n)$, leading to the following series of values :
$$0,\ 1,\ 1,\ 0,\ -1,\ -1,\ 0,\ 1,\ 1,\ 0, \ldots .$$
This gives $\pi = 3$.

J. T.

2055. *Definitions of limits and derivatives.*

The purpose of this note is to point out fundamental differences in the definitions by leading authorities, but not to discuss their merits.

Hardy, *Pure Mathematics*, Art. 93, defines a function which tends to a limit, or to ∞, or to $-\infty$. Hence, when he says that a function tends to a limit, he means that this limit is *finite*. Consequently, when he comes to define a derivative in Art. 110 he requires $\{\phi(x+h) - \phi(x)\}/h$ to " tend to a

limit" when h tends to zero, so excluding the case of an infinite derivative. He can then say (Art. 112) that "if $\phi(x)$ is not continuous for a value of x, then it cannot possibly have a derivative for that value of x".

De la Vallée Poussin, *Cours d'Analyse*, Art. 14, defines "tending to a limit" in such a way as to include $\pm\infty$. When he comes to define a derivative in Art. 49, he says: "the limit, finite or infinite, as $h \to 0$ of $\{f(x+h) - f(x)\}/h$". In the same article he proves the theorem: "Any function which has a *finite* derivative for a given value of x is continuous at this point". He does not give an example of a function which is discontinuous at a point but (according to his definition) has a derivative there. Such a function is the step-function, the sum of the Fourier series.

$$\sin x + \tfrac{1}{3} \sin 3x + \tfrac{1}{5} \sin 5x + \tfrac{1}{7} \sin 7x + \dots,$$

which is 0 when $x = 0$, $\pi/4$ when $0 < x < \pi$ and $-\pi/4$ when $-\pi < x < 0$. The function is discontinuous at 0 and yet has the derivative $+\infty$ there.

Young, *Fundamental Theorems of the Differential Calculus*, Art. 2, includes "$+\infty$ and $-\infty$ as among the values which the most general kind of function may assume". In Art. 9 he writes: "At a point at which $f(x)$ is not finite, or is discontinuous, we shall say that a differential coefficient does not exist". He goes on to say: "It follows from the definition that a differential coefficient is not necessarily finite, it may have the value $+\infty$ or $-\infty$ at particular points". He therefore differs from de la Vallée Poussin, and would not admit that the above-mentioned function has a derivative at $x = 0$. He does not have to prove that a function which has a finite derivative is continuous, since he makes this a matter of definition.

The general practice in this country seems to be to write $\lim f(x) = l$ or $\lim f(x) = \infty$, contrary to Hardy's definitions. Rolle's theorem takes on a different meaning according to the previous definition of the derivative, so that de la Vallée Poussin's statement of the theorem is slightly more general than Hardy's.

2056. *On partial fractions.*

The following method is doubtless far from new, but it does not seem to be given the place it deserves in the textbooks. It determines the numerator of a partial fraction with a quadratic denominator whose factors are surd or complex. I give one example of each type.

1. To find B and C if

$$\frac{x^2 - 3x + 1}{(x-3)(x^2+4x+1)} \equiv \frac{A}{x-3} + \frac{Bx + C}{x^2 + 4x + 1}.$$

Then, *for the value* $x = 2 + \sqrt{3}$,

$$Bx + C = (x^2 - 3x + 1)/(x - 3),$$

that is,

$$B(2 + \sqrt{3}) + C = (2 + \sqrt{3})/(-1 + \sqrt{3}),$$
$$= \tfrac{1}{2}(5 + 3\sqrt{3}),$$

whence $B = \frac{3}{2}$, $C = -\frac{1}{2}$.

2. To find B and C if

$$\frac{x^2}{(x-2)(x^2-2x+2)} \equiv \frac{A}{x-2} + \frac{Bx + C}{x^2 - 2x + 2}.$$

For the value $x = 1 + i$,

$$Bx + C = x^2/(x-2),$$

that is,

$$B + Bi + C = 2i/(-1 + i) = 1 - i,$$

whence

$$B = -1, \qquad C = 2.$$

H. V. Lowry.

REVIEWS.

Natural Philosophy of Cause and Chance. By MAX BORN. Pp. vii, 215. 17s. 6d. **1949.** (Geoffrey Cumberlege, Oxford University Press)

It may appear strange to describe a book of only 128 pages (which is all there are apart from appendices) as a comprehensive history of the present era of natural philosophy. But that is what Professor Born's Waynflete Lectures succeed in being in their published form. The content is history in that it treats the development of the subject, broadly speaking, in its historical sequence. But, although it ascribes the main advances to their authors with clear indications of their essential lines of thought, it is not concerned with purely historical questions for their own sake. It is comprehensive in that all the major advances are brought under review. But its concern is only with major advances.

Our description of the book need not appear inappropriate when regard is had to the aim of the work and to the skill of the writer. For a treatment which aims at presenting a perfectly clear picture of the march of ideas must be concise in order that the reader may assimilate the picture in its entirety. Details are better left to be supplied by supplementary sketches, and this explains the rôle of the appendices which occupy another eighty-odd pages.

Only a master, in knowledge and in craftsmanship, can produce such a picture. Professor Born has undoubtedly achieved a masterpiece.

One essential feature of any masterpiece is that it should possess a unifying theme. In this case the theme is *causality*. Certain outstanding advances are treated partly for their own special interest, as well as for their bearing upon the theme. But the theme remains clearly in view throughout the work.

Born's definition is :

" *Causality* postulates that there are laws by which the occurrence of an entity B of a certain class depends on the occurrence of an entity A of another class, where the word ' entity ' means any physical object, phenomenon, situation, or event. A is called the cause, B the effect."

To this he adds : " If causality refers to single events, the following attributes of causality must be considered :

" *Antecedence* postulates that the cause must be prior to, or at least simultaneous with, the effect.

" *Contiguity* postulates that cause and effect must be in spatial contact or connected by a chain of intermediate things in contact."

As I understand his approach, it is that science would be impossible without some acceptance of causality (his definition showing what scientists mean by the term), but that the whole question of its status and scope in physical science has to be fully investigated. The two attributes he defines are ones which it is generally regarded as " repugnant " to deny. I do not take Born to imply that they are necessary attributes, but merely that he proposes to examine the extent to which they are possessed by the causality to be discerned in the laws of physics.

Born reviews Newtonian particle mechanics and gravitation, which are completely causal systems, but which do not satisfy the principles of antecedence and contiguity. He then examines the degree of satisfaction of contiguity by classical mechanics of continuous media. He proceeds to show that it is fully satisfied by the mechanics and gravitation of Einstein's general relativity. He then shows that in thermodynamics antecedence is satisfied, but only at the sacrifice of any detailed analysis of the processes by which a system passes from one state of thermodynamic equilibrium to another.

Further advance depends upon the introduction of atomistics and statistics. This is where *chance* first comes into physical theories. But Born reminds us that chance enters into scientific activity at a much more primitive stage in connection with errors of observation and induction from experience. He therefore considers that " chance is a more fundamental conception than causality ".

Born goes on to show how probability is incorporated by postulates and by the consequent mathematical formulation in kinetic theory and statistical mechanics. He examines in particular " the metamorphosis of reversible mechanics into irreversible thermodynamics with the help of probability ". This, as he says, is brought about by " a deliberate renunciation of the demand that in principle the fate of every single particle can be determined ". He shows how, even on the ground of classical mechanics, this renunciation is practically inevitable.

The general conclusion that the renunciation, *i.e.* the admission of some degree of ignorance and the consequent resort to probability, does lead to irreversibility and hence to the statistical foundation of thermodynamics is quite familiar. But Born's analysis of the way it actually operates is a triumph of exposition in an exceedingly difficult domain.

It is also well known that the next step is the replacement of " a deliberate renunciation " by the recognition of uncertainty inherent in what can in fact be known about physical systems. This is at the basis of quantum mechanics and the rest of the work deals with this theory. Here again Born gives a display of unrivalled expository power in his explanation both of the inductive foundations of the theory and of the way in which its mathematical formulation rests upon these foundations. He shows that in quantum physics we still treat causal dependence, but that the dependence is not between single elementary events but the probabilities of those events. He concludes: " . . . chance has become the primary notion, mechanics an expression of its quantitative laws, and the overwhelming evidence of causality with all its attributes in the realm of ordinary experience is satisfactorily explained by the statistical laws of large numbers."

The foregoing summary may give the impression that Professor Born has taken us over very familiar ground. But what he says in effect is, We know what physical theories are in existence ; let us now go back over the ground to try to understand the predominant trends of the thought which has gone into their making ; in particular, let us see what has happened to our fundamental notions of cause and effect and, even more particularly, to our belief that the cause must precede its effect and must be in the same place as its effect.

In the worthy pursuit of their calling, the metaphysicians may inform us that a physical law is " really " a deductive theorem or that it is " really " a mere description of perceptions. But their conclusions, whatever they are, are remote from the whole mode of thinking of the practising physicists who have actually established the law through their interpretation of experience. The principles of the physicists' thinking are what constitute Natural Philosophy. It is these that Professor Born is concerned to elucidate. He has written a classic of Natural Philosophy. W. H. McCrea.

Length and Area. By Tibor Radó. Pp. v, 572. $6.75. 1948. American Mathematical Society Colloquium Publications. Vol. XXX. (American Mathematical Society, New York)

Professor Radó here presents what he describes as the " analytic " theory of length and area : the theory initiated by Lebesgue nearly fifty years ago by his definition of surface area in terms of the areas of polyhedra which tend

uniformly to the surface. He is not directly concerned with the " measure-theoretic " ideas of length and area : that is to say, he deals with curves and surfaces as represented by vector functions of real variables and with the invariance of length and area under a change of representation, and not with the one- or two-dimensional measure of curves and surfaces regarded merely as sets of points without such representation. Consequently there are only passing references to the work of Besicovitch and others in this latter field. But even so, the task of giving a unified presentation of the work of Lebesgue, Geöcze, and many others, including the important contributions of the author himself, was formidable. Professor Radó deserves the most grateful thanks of all interested in the subject, not only for undertaking the work at all, but for maintaining an almost miraculous lucidity (considering the inherent difficulties of the problems) even up to the five-hundredth page.

He begins, after explaining in a short introduction the plan of the book and some of the problems involved, with two useful chapters entitled " Background in Topology " and " Background in Analysis ", where he assembles some of the weapons required in later chapters : for example, theorems concerning the topology of the Euclidean plane and the two-sphere, and the fundamental properties of the Lebesgue integral. Then follow four sections in which definitions and proofs are given in full detail. Section II (Curves and Surfaces) is devoted mainly to topological ideas : the idea of Fréchet equivalence and the definition of a path-curve or path-surface in terms of equivalence classes of representations, cyclic additivity, and the topological index. Section III is entitled " Arc Length and Related Topics ". Here the ideas are mainly analytical : interval functions and their completely additive extensions, Burkill integrals, variation and absolute continuity, leading up to a discussion of arc-length, Steiner's inequality, and related topics. Section IV (Plane Transformations) is concerned with maps of plane Jordan regions into a plane, generalised Jacobians and the transformation of integrals. Finally Section V (Surface Area) deals with a " lower area " $a(s)$ defined in terms of integrals of " essential multiplicity functions " and with the Lebesgue area $A(s)$, and proves the equality of $a(s)$ and $A(s)$ when $A(s)$ is finite or $a(s)$ is zero : the author points out in a later note that the complete equivalence of $a(s)$ and $A(s)$ was proved by L. Cesari in Italy during the war. The last chapter is devoted to surfaces defined in the form $z=f(x, y)$, which can be treated by methods more nearly akin to those applicable to curves, but are here considered rather as special cases of the more general representations.

At the end of each of the four main sections the author has inserted a chapter which explains, with as few technicalities as possible, the main ideas of the preceding chapters, their history, and their relevance to the general theory he is developing ; he also discusses some of the unsolved problems. These little informal lectures to the reader are most valuable. Without them the book would be an indispensable work of reference, but somewhat heavy going for those unfamiliar with the subject ; these fifty or sixty pages of exposition lighten the whole book and make it not merely learned but attractive and readable. U. S. H.-J.

Set Functions. By Hans Hahn and Arthur Rosenthal. Pp. ix, 324. $12. 1948. (University of New Mexico Press)

Although Professor Hahn published two successive and quite distinct editions of the first volume of his treatise on real functions (*Theorie der reelle Funktionen*, I, Berlin, 1921, and *Reelle Funktionen*, Leipzig, 1932), the second volume was still unfinished at the time of his death in 1937. Professor Rosenthal generously undertook the task of editing and completing the manuscript, and by 1942 the German text of this volume was ready for press.

But by that time conditions in Europe made its publication impossible, and therefore he has written a new version in English which includes the greater part of Hahn's original manuscript, but omits certain sections which were concerned more particularly with point functions than with set functions.

The scope of the book is, roughly speaking, that of the first half of Saks's *Theory of the Integral*, but the subject is treated in much greater detail. After a short introduction concerning the elementary topology of sets in general spaces, Chapter I deals with additive and totally (*i.e.* completely) additive set functions, including (in generalised terms) such topics as the Lebesgue decomposition into regular (absolutely continuous) and singular parts. After a discussion of Carathéodory measure in Chapter II, Euclidean space R^n is introduced and the previous abstract theorems are specialised in this space. Chapter III concerns measurable point functions taking real values. The integral of a point function with respect to a totally additive set function is defined in Chapter IV (roughly speaking, as a totally additive set function satisfying a mean-value condition), and this is connected with Lebesgue, Riemann and Darboux sums. The second half of the chapter deals with mean-value theorems, convergence in mean p-th power, and finally with product spaces, Fubini's theorem and repeated integrals. The last chapter, on derivates of set functions, begins by considering systems of covering sets of Vitali and similar types, and derivates with respect to these systems, and applies the results to metric density, approximate continuity, and the relation between derivates and integrals : finally, there is a short section on interval functions and their related set functions.

As in Hahn's earlier volumes, all the definitions, enunciations and proofs are set out in minute detail (occasionally perhaps too minute for the comfort of the reader). There are probably over a thousand theorems in the book, and it is a pity that the printing or the system of enumeration does not distinguish between the theorems which are required for later use and the lemmas which have only an immediate value. For this reason, and because the whole plan of the work is to prove all that is possible in the most general space before proceeding to the less general, it is not, as Professor Rosenthal says in his preface, a textbook for beginners. But he has done much to lighten the reader's task by introducing in footnotes, particularly in the earlier part of the book, simple and illuminating examples to emphasise the essential character of the conditions imposed on his theorems.

The book will undoubtedly be useful to those working or preparing to work on set functions and allied topics, for there is much in it that is not readily available elsewhere, and there are many valuable bibliographies scattered throughout the chapters. U. S. H.-J.

Leçons sur le calcul des coefficients d'une série trigonométrique ; quatrième partie : les totalisations, solution du problème de Fourier. By A. Denjoy. Two volumes. Pp. 327–715. 3700 fr. 1949. (Gauthier-Villars)

This important work, in which the author sets out in full and improved form his famous theory of " totalisation " and illustrates its power by solving the classical " problem of Fourier ", is now complete with the publication of its fourth part, itself divided into two volumes.

A real function $f(x)$ of period 2π is given which, it is assumed, can be developed into an everywhere convergent trigonometrical series

$$f(x)=\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx). \quad \text{.....................(1)}$$

There are two questions which arise immediately. The first concerns the uniqueness of the development ; the answer, given by Cantor, is affirmative.

The second question is the " problem of Fourier " : if $f(x)$, known as developable, is given, how can the coefficients of (1) be determined? The classical answer is contained in the formulae

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx \, dx, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx \, dx, \quad \ldots\ldots\ldots\ldots(2)$$

given by Euler and Fourier : the series (1) is the " Fourier series " of $f(x)$. This answer presupposes the notion of integral employed in (2). In the case of the Lebesgue integral, (2) holds whenever $f(x)$ (developable) is finite and integrable (summable) over $< 0, 2\pi >$ [Du Bois-Reymond and Lebesgue]. There exist, however, developable functions such as the sum of $\sum_{n=1}^{\infty} \frac{\sin nx}{\log (n+1)}$, which are not summable. In some such cases a generalisation of the notion of integral, a " total " in the vocabulary of the author, allows one to re-establish the formulae (2) in a wider sense. There exist, however, developable functions where even totalisation does not yield such simple result. In the most general case, Riemann's method of considering the twice integrated series (1), that is,

$$F(x) = \tfrac{1}{4}a_0x^2 + Cx + C' - \sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{n^2} \quad \ldots\ldots\ldots\ldots(3)$$

is employed. Since a_n and b_n tend to 0 [Cantor], the series (3) is uniformly convergent and $F(x)$ is continuous. Conversely

$$\frac{F(x+h) - 2F(x) + F(x-h)}{h^2} \to f(x) \quad (h \to 0) \quad \ldots\ldots\ldots\ldots(4)$$

for all x : $f(x)$ is the " generalised second derivative " of $F(x)$. Given such an $f(x)$, the problem is to find its " generalised second primitive " $F(x)$, determined apart from a linear term $(Cx + C')$. Once $F(x)$ is known, the coefficient c_0 of (1) is obtained as $\lim_{x\to\infty} \frac{4F(x)}{x^2}$, while $-\frac{a_n}{n^2}$, $-\frac{b_n}{n^2}$ are, for $n \geqslant 1$, the Fourier coefficients of the continuous periodic part of $F(x)$: the problem of Fourier is solved.

This solution of the problem, the actual determination of $F(x)$ by repeated processes of totalisation, is the main theme of the present work, as is apparent from its title. Its real importance, however, lies in the lucid exposition and thorough analysis of the process of totalisation itself. Roughly speaking, the problem of totalisation is the determination of an indefinite (non-absolute) integral, or primitive function, of a non-summable function. It is impossible to do justice, in a short and semi-popular review, to the depth of the ideas and the richness of the material presented here. This is a book for the serious student of the modern methods of real analysis, who will find its attentive study indispensable. It is written in a masterly personal manner. The author, who confesses himself rather a pupil of Baire than of Lebesgue, lays much stress on the " constructive " definition of his totals as against, for instance, the " postulated " existence of the so-called Perron integral. Although he is certainly right there in principle, the reviewer must confess that he cannot see how such constructive features (in the usual sense of the word) can be attributed to the most general processes of totalisation where transfinite induction (into the second class of ordinals) is required.

W. W. Rogosinski.

Number Theory and its History. By Oystein Ore. Pp. x, 370. 27s. 1948. (McGraw-Hill)

Until fairly recently there were very few works in English which gave an attractive account of elementary number theory. The gap was filled to a

large extent by Hardy and Wright's *Introduction to the Theory of Numbers* and Uspensky and Heaslet's *Elementary Number Theory*. Professor Ore's book is another welcome contribution to the literature of the subject and offers yet a third approach.

It is hardly possible to give a satisfactory account of elementary number theory without making frequent references to the history of the subject. Some results can be traced back to remote ages, others seem only to have been discovered in recent times, often by gifted amateurs. In the present volume the author has interwoven the history with the subject-matter, and has produced a book which should have a wide appeal, not restricted to professional mathematicians or to serious students of mathematics.

As regards the mathematical content, the ground covered is considerably less than in either of the two works mentioned earlier. There is practically nothing about quadratic residues, for instance. Probably the deepest result that is proved is the existence of a primitive root to any prime modulus, or prime power modulus. But the historical material is unusually rich, and ranges in time from the Old Babylonian period to the life of Gauss.

Probably every reader will find something that is new to him. The reviewer must confess that he was ignorant of the very simple method of " Russian multiplication " (page 88), and of Lawther's application of indices to the splicing of telephone cables (pages 305–310).

There is much in the book which will be of interest to every educated person, even if his knowledge of mathematics is rudimentary. But such a reader may find the transition from the historical material to the mathematical symbolism a little abrupt, and perhaps a little more might have been done to help him. H. D.

1. **Vectorial Mechanics.** By E. A. Milne. Pp. xiii, 382. 35s. 1948. (London : Methuen)

2. **Vector and Tensor Analysis.** By Louis Brand. Pp. xvi, 439. 33s. 1947. (New York : John Wiley and Sons ; London : Chapman and Hall)

Although many books have been published in recent years in which vector and tensor methods are used for solving problems in geometry and mathematical physics, there has been a lack of first-class treatises which explain the methods in full detail and are nevertheless suitable for the undergraduate student. In applied mathematics no book has appeared till now which is comparable with Hardy's *Pure Mathematics*. This deficiency is the more surprising when we recall that the rigorous analysis of limiting processes was continental in origin, whereas the vector and tensor techniques have been due in no small part to Anglo-Saxon mathematicians. The reason for this neglect may be that the applied mathematician usually regards mathematics as a means to an end, whilst the pure mathematician looks upon mathematics as an end in itself. Whereas the former uses mathematical techniques merely as intellectual tools supplementing the methods of observation and experiment, the latter is more interested in their logical and aesthetic critique.

It is well known that pure and applied mathematicians tend to disagree about the merits of the vector calculus of Gibbs. The pure mathematician regards the quaternion as a more significant generalisation of the number concept than the Gibbsian vector. The number concept is defined by specific laws of addition, multiplication, etc. If we relax the commutative law of multiplication of ordinary arithmetic, but retain the law that the product of two non-zero numbers can never be zero, quaternions are found to be the resulting generalised numbers. Vector algebra, however, does not arise as a similar automatic " structural " generalisation of classical algebra. Never-

theless, the applied mathematician has found that in practice vectors are more useful than quaternions. Unfortunately, he has seldom troubled to give the same meticulous attention to the development of the vector idea which the analyst has devoted to the number concept.

The vector idea is the common element in a number of physical concepts which involve the notion of direction. A vector is not a generalised number, in the sense in which a quaternion is, any more than the concept of mass is merely a generalisation of the concept of weight. It involves a new idea. This is the key-note of Professor Milne's long, and eagerly, awaited treatise on *Vectorial Mechanics*. As he remarks in the preface, its genesis dates back to about 1924 when Professor Chapman first expounded to him the view that "vectors were not merely a pretty toy, suitable only for elegant proofs of general theorems, but were a powerful weapon of workaday mathematical investigation, both in research and in solving problems of the types set in English examinations ". As Milne goes on to remark : " I did not at first believe him ; I had been brought up in the idea that in the words of a distinguished applied mathematician, vectors were like a pocket-rule, which needs to be unfolded before it can be applied and used ; similarly I thought (with most others at that time) that vector expressions were a mere shorthand for sets of Cartesian expressions, and that before they could be interpreted they always needed to be translated into Cartesians." Under Chapman's inspiration, however, Milne soon found that " problems, dull or difficult by conventional methods, gained when treated by vector methods an interest, an ease and a delight which they previously lacked ; and that when a problem was formulated and solved vectorially, the vector solution provided a kinematic picture of the motion in question that gave far more insight into the phenomenon than the corresponding Cartesian solution. The Cartesian solution tells *where a system is* ; the vector solution, with one less integration, tells *how it is moving* ".

Plans were soon made for a joint book on vector methods in applied mathematics, based on lectures given by Milne at Oxford and by Chapman at the Imperial College. Unfortunately, only Milne actually completed his draft, which was confined to kinematics, statics and dynamics ; the joint plan was to have included other subjects, for example, electricity and magnetism. Although the manuscript was finished in 1938, war-work delayed the final revision, and, to complete the story, the process of publication seems to have lasted for over three years, which must be a record even in these days!

Most of the theorems in this book are standard, but Milne claims with justice to have exhibited for the first time the tremendous power latent in the process of vector multiplication. " The theme of the book is in fact the vector product $\mathbf{A} \wedge \mathbf{B}$ and its applications." This symbol for the product is due to Chapman, Milne and Hartree, the use of brackets being reduced thereby to a minimum. The ease and elegance of this notation is responsible in no small degree for the feeling of power which the vector calculus now provides.

The book is divided into three parts : Vector Algebra, Systems of Line-Vectors, and Dynamics. The last comprises more than half the book, and is the most thorough vectorial treatment of the subject which has yet appeared. The first two parts of the book include discussions of statics and finite displacement of rigid bodies, but are primarily concerned with the *theory* of vectors. Just as in Hardy's classic, a new note is struck at the very start : a precise definition is given of the concept " free vector ", analogous to the Frege-Russell definition of " cardinal number ". According to Milne, a free vector is the class of all its representations, a typical representation being defined in the customary manner. From a pedagogic point of view, however, the reviewer wonders whether it might have been better to draw attention at

this early stage to a concrete instance of a *free* vector. The student familiar with physical concepts which have magnitude and position, but not direction, should be made to realise from the very beginning that the free vector is not merely " fundamental in discussing systems of position vectors and systems of line-vectors ", but occurs naturally in its own right, as there are physical concepts which have magnitude and direction but not position, *e.g.* the couple in statics, and the angular velocity of a rigid body. Although the necessary existence theorems must be established at a later stage, and Milne's rigorous proofs are particularly welcome, there is no reason why some instances of free vectors should not be mentioned at this point.

The essence of the vector concept is the idea of direction. Although this is clearly emphasised by Milne, he makes no use of Heaviside's convenient notation $\check{\mathbf{a}}$ for the " ort ", or unit vector, in the direction of the free vector $\mathbf{a}$. With this symbol we can often avoid clumsy expressions of the type $\mathbf{a}/|\mathbf{a}|$. Heaviside's symbol also replaces the cumbersome triad of direction cosines of analytical geometry, and materially assists in simplifying the proofs of many theorems, *e.g.* the important continued (or triple) vector product theorem of which Milne gives three independent proofs.

Despite the authoritative and thorough development of the ideas, technique and symbolism of vector algebra and calculus in the first two parts, perhaps the outstanding feature of this book is the skill with which in the third part vector methods are adapted to the solution of kinematical and dynamical problems in three dimensions. A beautiful example arises in the problem of the motion of a rough sphere pressed between two parallel plane boards rotating with different angular velocities about non-coincident axes orthogonal to themselves. The power of the vector technique is particularly apparent in its application to the theory of motion in rotating frames of reference, and the treatment of this subject is one of the most fascinating features of the book. Especially welcome is the inclusion of Larmor's theorem relating to the precession of electronic orbits in a magnetic field. This is one instance among many in which the author illuminates the subject by showing that dynamics is truly the theory of matter in motion, and not merely a traditional part of a routine examination syllabus.

The distinctive peculiarity of vector analysis is the construction of the vector product. This technique, however, is strictly appropriate only to problems in *three* dimensions, where it enables us to dispense with the artificial and cumbersome Cartesian triad of orthogonal axes. On the whole, the Gibbsian method is much less useful when dealing with two-dimensional systems, although the ideas of vector addition and scalar product are still valuable. The appropriate vector technique for problems in the plane is that of complex numbers, but in the theory of orbits, for example, it is difficult to improve upon classical scalar methods. Nevertheless, Milne's discussion of orbit theory, including Hamilton's transformation and integral, makes a refreshing change from the traditional line of approach. When, as in the case of Lagrange's equations, however, we consider n-dimensional manifolds, the tensor technique is required. Tensor methods have the great advantage, moreover, that they are not restricted to spaces with a Euclidean metric. Milne develops both the theory and the applications of *Cartesian* tensors and dyads with great elegance, but it is open to question whether the more elaborate parts of this technique will appeal equally to all students in view of the increasing popularity of the general tensor calculus.

This calculus is described in Professor Brand's new book, *Vector and Tensor Analysis*, although his discussion might with profit have been expanded at the expense of his unnecessarily long-winded account of vector analysis. The latter is treated in a less fundamental manner than in Milne's book. On the

other hand, Brand provides a greater selection of unworked examples. His discussion of the vector triple product formula is very scrappy and compares unfavourably with the detailed account given by Milne. The most curious feature of the book is the chapter on line-vectors. The author introduces a new technique for manipulating dual-vectors,

$$\mathbf{F} = \mathbf{f} + \epsilon \mathbf{f}_O,$$

where ϵ is an algebraic unit whose square is zero ; $\mathbf{f}$ is called the resultant vector and is independent of any particular choice of origin, and $\mathbf{f}_O$ is called the moment vector which changes with change of origin O to P according to the law

$$\mathbf{f}_P = \mathbf{f}_O - \mathbf{PO} \wedge \mathbf{f}.$$

A line-vector is a dual-vector for which the scalar product of $\mathbf{f}$ and $\mathbf{f}_O$ vanishes. This treatment is both novel and unnecessary ; it adds nothing significant to more familiar methods and is definitely not to be recommended to beginners. Probably because of an implicit reference to Clifford, this technique is called " motor algebra ".

The chapter on " Differential Invariants " is fairly thorough, although it is unfortunate that the author employs the term *rot* rather than *curl*. An extremely full selection of examples is included. It is followed by a detailed chapter on " Integral Transformations ". Perhaps the most useful chapter in the book, however, is that on " Tensor Analysis ", but it should have come much earlier. Brand first develops this analysis for Cartesian frames in 3-space and then for general co-ordinate systems in n-space. The chapter concludes with a short introduction to Riemannian geometry, although the main emphasis is on ordinary space. Unfortunately, the author persists in using unorthodox terminology, speaking of the " valence " of a tensor when he means " rank ".

Both books are well and clearly printed, although Milne's does not conform to the usual convention of printing scalar symbols in italics. Brand's book is provided with an index. There is no doubt that Milne's book will become a standard treatise on dynamics, the most important to appear in this country since Whittaker's. Brand's book requires careful pruning before it can be recommended to the novice. As it stands it contains much unnecessary material.

G. J. W.

Vektorrechnung. By H. Athen. Pp. 90. DM. 6.50. 1948. (Wolfenbütteler Verlagsanstalt)

On p. 269 of the *Gazette* (October, 1948) a book is reviewed which is one of a series produced in Germany since the war to attempt to meet the current shortage of scientific textbooks in that country. The present volume is another of the same series. Like the first book, it offers " austerity " in appearance and production, but in this case at any rate the contents also are controlled largely by this principle. Since the series is intended primarily for the use of students, the author has deliberately restricted himself to the use of established methods of presenting his subject. The general reader must therefore expect a sound conventional treatment rather than novelties. Nevertheless, the introduction and application of vector methods are considered from a number of different points of view. The author has kept to a minimum the discussion of applications to pure and applied mathematics and to physics, at the same time promising to deal with these in a supplementary volume of the same series. In spite of this, the most impressive feature of the book is the amount of material he manages to include in the 88 pages of text.

The book is divided into four main chapters. In the first of these, vectors

are defined and vector algebra introduced. More use is made of coordinates here, and indeed throughout the book, than is usual nowadays, but geometrical and operational methods are not overlooked by any means. In the second chapter a comprehensive treatment is given of the theory of the operators *div*, *grad* and *curl*. Each of these is introduced with great care by means of a physical analogy. I feel that the author was less successful with the last of these than with the other two.

The third chapter is devoted to the structure of vector fields, including general potential theory, while the last chapter covers tensors of the second rank. An appendix gives solutions of twelve examples which are scattered throughout the book.

In conclusion, one can say that the book is competently written and should be very successful for the purpose for which it was intended. B. M. B.

Kurvenintegrale und Begründung der Funktionentheorie. By LOTHAR HEFFTER. Pp. iv, 48. DM. 5.40. 1948. (Springer, Berlin)

The booklet contains material published by the author in the years 1902–1941, and aims to establish a new foundation of the theory of functions. Chapter I contains necessary elements of the real variable theory, defining Riemann integration. Chapter II deals with rectifiable continuous curves. Chapter III contains definition and discussion of the curvilinear integral $\int(f\,.\,dx+g\,.\,dy)$. Chapter IV discusses the connection between a curvilinear integral and a Stieltjes integral. Chapters V and VI contain the principal material. Property (EJ) is defined: *For every rectangle R, in a closed connected region G, of corners $a+i\alpha$, $b+i\alpha$, $b+i\beta$, $a+i\beta$, the integral $\int_R f(z)\,dz=0$.*

THEOREM 32: *If $f(z)\equiv u(x,y)+iv(x,y)$ is integrable in G with respect to x for every y and to y for every x, then (EJ) is equivalent to*

$$\left.\begin{aligned}\frac{u_{ab}(x,\alpha)-u_{ab}(x,\beta)}{\alpha-\beta}&=-\frac{v_{\alpha\beta}(a,y)-v_{\alpha\beta}(b,y)}{a-b}\\ \frac{v_{ab}(x,\alpha)-v_{ab}(x,\beta)}{\alpha-\beta}&=\frac{u_{\alpha\beta}(a,y)-u_{\alpha\beta}(b,y)}{a-b}\end{aligned}\right\}, \qquad\qquad(12)$$

where $u_{ab}(x,\alpha)$ denotes the integral mean of $u(x,\alpha)$ in $a\leqslant x\leqslant b$.

THEOREM 33: *If $f(z)$ has the property (EJ), and is continuous in G, and if $h(z)$ is continuously differentiable in G, satisfying the Cauchy conditions, then $\int_R f(z)h(z)dz=0$.*

Taking $h(t)=1/(t-z)$, Cauchy's integral formula follows on the assumptions (12) for $f(z)$ only. These are weaker than Goursat's conditions, since the existence of a derivative is not assumed.

Chapter VII contains a list of relevant original papers and some polemic material concerning the merits of the method and a critical remark of L. Bieberbach. P. VERMES.

Mathematics in Aristotle. By Sir THOMAS HEATH. Pp. xiv, 292. 21s. 1949. (Oxford University Press)

This book, finished shortly before its author's death and prepared for publication by his widow, forms a fitting climax to Sir Thomas Heath's great life-work in the study of the mathematics of ancient Greece. "The importance of a proper understanding of the mathematics in Aristotle," he tells us in his Introduction, "lies principally in the fact that most of his illustrations of scientific method are taken from mathematics."

The method followed is admirable. After a short preliminary section on Aristotle's view of mathematics in relation to the sciences, we are given a translation with commentary of all the mathematical passages in Aristotle,

arranged, with full references, under the names of the works in which they occur. As a display of the state of mathematical knowledge in the fourth century B.C., this compendium is highly instructive, but the modern reader will find Heath's comments and discussions at least as interesting as the texts themselves, especially when he applies himself to the unravelling of confusions and obscurities. Where everything is so illuminating, discrimination is invidious, but it may perhaps be relevant to draw the reader's attention specially to the passages dealing with incommensurables, the infinite, falling bodies and the paradoxes of Zeno. This is pre-eminently a book to be browsed in, and every reader will find in it his own speaial gems.

Hardy pointed out some time ago that mathematics is, above all other studies, a young man's special field ; it is interesting to find that Aristotle agrees, for " the subjects of mathematics are reached by means of abstraction, while the principles of philosophy and physics come from experience ". And how rigorous are Aristotle's demands is made clear when he rather cuttingly remarks that " it seems just as wrong to accept merely plausible arguments from a mathematician as to require demonstrations from a rhetorician."

E. L. MASCALL.

Elementary Calculus and Coordinate Geometry. II. By C. G. NOBBS. Pp. 399. 17s. 6d. 1949. (Oxford University Press)

The five chapters in this second volume are :

VI. Differentiation and Integration.
VII. Further Applications to Curves.
VIII. Miscellaneous Coordinate Geometry.
IX. Functions.
X. Applications and Numerical Calculations.

In Ch. VI the rules for differentiating uv, u/v, y^2 (with respect to x) are given, with indications of the method of proving them ; the function of a function rule is examined in some detail so that the pupil brought up on differentials will not be left with the idea that no proof is required ; " rates of change " problems are solved, and the usual " applications to mechanics " dealt with. Finally, the definition of an integral (by summation) is examined at length and some elementary properties of integrals, such as

$$\int_a^b = \int_a^c + \int_c^b, \quad \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

are obtained.

Ch. VII introduces curves given by parametric equations, polar coordinates, the formula $p = \pm(ax+by+c)/\sqrt{(a^2+b^2)}$. Some preliminary investigations of the " standard " conics follow. Further elementary analytical geometry formulae appear in Ch. VIII, and a more detailed account of the second degree equation is given.

Ch. IX begins with examples of functions defined by different formulae in different intervals, and the author goes on to explain (as far as is possible in very elementary terms) what is meant by continuity and differentiability. It should be pointed out that the solution given on p. 200 is incorrect, since if $f(x)$ is differentiable it is then necessarily continuous ; the two restrictions given for continuity and differentiability should have been successively imposed in that order. The rule $d(x^n)/dx = nx^{n-1}$ is proved for the usual values of n and then for $n = \sqrt{2}$. Here the author is, I think, on dangerous ground. However right he is in this case, I feel that an argument which even suggests that because

$$x^{1\cdot41} < x^{\sqrt{2}} < x^{1\cdot42},$$

it is *reasonable* to suppose that

$$1{\cdot}41x^{0{\cdot}41} < \frac{d}{dx}(x^{\sqrt{2}}) < 1{\cdot}42x^{0{\cdot}42}$$

is most undesirable.

Circular measure is next introduced, and the derivatives of sin x, cos x and tan x are obtained. Mention is made of inverse and implicit functions, and the reader is shown how to find the axes of $ax^2+2hxy+by^2=1$ by making $r^2=x^2+y^2$ a maximum or minimum.

The book concludes with a most interesting chapter on approximations. By way of introduction the binomial theorem is obtained (for n a positive integer) by differentiating

$$(1+x)^n=\Sigma a_r x^r,$$

and the same method is applied to obtain the results for $(1+x)^n$ when n is not a positive integer, and for sin x and cos x. That assumptions have to be made to allow of differentiation is carefully pointed out, and with these results to illustrate, the author explains what are meant by first, second, ... approximations. A paragraph on location of roots follows, and then the various methods of solving equations approximately are introduced and given detailed treatment. First we have " successive approximation " ; if a_1 is an approximation to a root of $f(x)=0$, put a_1+h for x and neglect h^2, ... This gives an approximate value for h, and the work is repeated with a_1+h replacing a_1. This method and the next (" proportional parts ") are fairly well known. Thirdly, we are shown Newton's method, which yields the iteration

$$a_{n+1}=a_n-f(a_n)/f'(a_n),$$

and the author goes on to show that the required root can often be more easily obtained by using the iteration

$$a_{n+1}=a_n-f(a_n)/f'(a_1),$$

or even

$$a_{n+1}=a_n-f(a_n)/g,$$

where g is a convenient constant nearly equal to $f'(a_1)$. Finally, it is shown that the iteration

$$x_{n+1}=\phi(x_n)$$

usually gives closer and closer approximations to a root of the equation $x=\phi(x)$ as n increases, if x_1 is taken near the root and $|\phi'(x_1)|<1$, and Mr. Nobbs illustrates by finding an approximation to the root near -1 of $x^3+10x+12=0$, with $\phi(x)=-(x^3+12)/10$, and to the root near 3 of $x^3-10x+1=0$ (which requires some modification, since $\phi'(3)>1$). The whole of this chapter is very detailed, and no effort is spared in an attempt to explain how to do a practical job efficiently. The chapter concludes with an account of the familiar methods of approximate integration.

The book contains a very large number of examples, carefully graded to suit pupils of different abilities.

e^x and log x are excluded from both Volumes I and II, which is a pity, I think, in a book running to some 650 pages. The omission imposes heavy limitations ; for example, integration is confined to $(ax+b)^n$, sin ax and an occasional $f'(x)/f(x)$. The book should be intelligible to the pupil without much assistance from a teacher, which is a great recommendation. R. W.

Introduction to the Theory of Finite Groups. By W. Ledermann. Pp. viii, 152. 8s. 6d. 1949. University Mathematical Texts. (Oliver and Boyd)

Problems of group-theory arise suddenly and unexpectedly in widely different branches of mathematics. To mention a few, in determinants, theory

of equations, elliptic functions, automorphic functions, invariant theory, relativity, quantum theory, the concepts of groups arise urgently and fundamentally. It is to be regretted, therefore, that in many mathematical courses group-theroy is completely ignored. It is true that in many instances where the need for groups would naturally arise, their use may be circumvented and equivalent results obtained without their mention. Recognition of the significance of groups, however, not only simplifies the separate theories, but also provides a binding link which has unifying influence on the whole mathematical structure.

One reason for the neglect of group-theory is undoubtedly the fact that the very few textbooks which deal with the subject have been designed for the specialist rather than for the general mathematician with other interests, and are too detailed and abstruse for the latter. This book should meet a real need. It gives an account of the fundamentals of group-theory that is sufficiently extensive to survey the principal aspects and yet sufficiently concise and readable for the non-specialist.

A lucid and detailed introduction is followed by a fairly thorough account of complexes and subgroups, culminating in Frobenius' decomposition of a group relative to two subgroups. There follow chapters on permutation groups and invariant subgroups. The latter contains a proof of the important Jordan-Hölder composition theorem, and also Galois' theorem that the alternating group for degree greater than 4 contains no invariant subgroup. One might have wished, however, that instead of the brief mention of the significance of this result in relation to the theory of equations there had been room for a short account of functions belonging to a group.

The last chapter concerns Sylow groups, prime power groups and Abelian groups.

If there is a fault to find it is perhaps that there is insufficient indication of the applications of group-theory, an aspect which should catch the attention of the non-specialist. Nevertheless, the book is a valuable addition to the library.

D. E. LITTLEWOOD.

Projective Geometry. By T. EWAN FAULKNER. Pp. viii, 128. 7s. 6d. 1949. (Oliver and Boyd)

A good book on projective geometry would have been a very welcome addition to the University Mathematical Texts. The book under review cannot be classed with the best of this well-known series. One has the feeling that the author has been out of touch with the geometrical world for some time. It is difficult to think of any Honours course in plane projective geometry for which one could recommend his book, although it has the great merit of cheapness.

Chapter I which, the Preface says, many readers may well find the most difficult, begins with foundations, and propositions of incidence are mentioned. The style is breezy ; that is, things are talked about, and not formulated precisely. But the first real shock is the introduction of " the symbolic notation " for a point, with no more explanation than is given in Vol. I of H. F. Baker's *Principles of Geometry*, to which the author refers. There need be no mystery in this very useful symbolic notation, if the equivalence of the symbol P and an ordered triad of elements of a field,

$$P=(x, y, z),$$

be postulated, so that P is an element of a vector space. If the foundations are to be algebraic, there can be no objection to this procedure. If the foundations are to be synthetic, one expects a logical development of the axioms of incidence until coordinates are introduced. The first chapter of

this book is a mysterious hash of both methods, and space is wasted on the "addition and multiplication of points on a line", when there is no conceivable reason for introducing these topics, since the author is not introducing coordinates by synthetic methods.

In Chapter III, which deals with the projective definition of a conic, the standard is much below that of Vol. II of Baker's *Principles of Geometry*. For example, no attempt is made to show that the vertices of the two projective pencils by which a conic is defined are not special points on the curve. Here, and in other chapters, far too much use is made of (1–1) correspondences, without any reference to the underlying algebra on which a rigorous use of (1–1) correspondences is based.

Chapters IV, V and VI deal with "Absolute Elements: the Circle: Foci of Conics", "The Equation of a Line and of a Conic: Algebraic Correspondence on a Conic: the Harmonic Locus and Envelope" and "Metrical Geometry".

The book is well produced and contains examples. It is with regret that the reviewer has decided that he cannot add it on his bookshelf to the two outstanding textbooks on projective geometry which have been produced in this country: Baker's *An Introduction to Plane Geometry* and Todd's *Projective and Analytical Geometry*. D. PEDOE.

Metodos para la Resolucion de los Problemas Geometricos. Por VICENTE INGLADA. Pp. 477. 110 pesetas. 1948. (Madrid: Editorial Dossat)

This book deals with a wide variety of geometrical topics in an interesting and suggestive manner. The first part gives an elementary account of geometrical transformations and their groups, passing on to consider in particular linear projective transformations, linear transformations in the complex plane, and then quadratic transformations in the projective plane. The treatment does not go very deep, but the striking feature of the book is the wealth of examples of a most varied kind, some of them perhaps unfamiliar to English readers. Thus his examples of transformations include "inversion" with regard to an ellipse, a triangle and a trihedron, contact transformations and conformal transformations; he works out the conditions that a pentagon should be transformable by a homology into a regular pentagon; he considers the projective transformation of a plane cubic into one of the "divergent parabolas" of Newton; he inverts a triangle into an equilateral triangle, applying the result to the solution of the cubic equation; he uses the quadratic transformation in connection with the Simson (Wallace) lines of a triangle, and to prove that a conic through the cusps of a three-cusped hypocycloid and the two points in which a tangent meets the curve again must be a rectangular hyperbola. It is all rather a hotch-potch, but it is great fun.

The second part is equally miscellaneous. There is first a collection of problems dealing "synthetically" with the determination of various loci and envelopes, and with maxima and minima. Then comes a section on the use of the complex variable in the theory of conics, for example, to prove that the centres of the five rectangular hyperbolas through the fours of five concyclic points lie on a circle; and finally, the author treats of vectors, with the familiar application to the differential geometry of curves and surfaces.

Many of the examples appear to have been set in examinations for entrance to Engineering Colleges in Spain; one doubts whether Tripos candidates would have been able to tackle them.

It will be clear from the above very brief account that Dr. Inglada's book is excellent to browse in, not only for its own sake, but also as a most suggestive source of examination questions. F. P. W.

Modern Elementary Geometry. By L. ROTH. Pp. 370. 8s. 6d. 1948. (Nelson)

According to the preface this is a course of formal geometry up to higher certificate stage in which more than the customary attention is paid to logical principles. The logical part is contained in sections headed BY THE WAY, and need not be taken at the first reading.

The author says that " as far as has seemed practicable the recommendations of the Mathematical Association's *Report on the Teaching of Geometry* have been followed ". This may refer to the first Report (1923), and it may be that the author has not seen the second Report (1938). Whether he has or not, it must be said that his book disregards recommendations of major importance made in both Reports about the Stage B deductive course.

As regards theorems, the book approximates to the contents of conservative textbooks published around 1903, though it is true that Euclid I.4 is replaced by a congruence axiom. After a short discussion of axioms and definitions we find : " To see what a theorem is like, we have only to examine a single specimen, Theorem 1 ", and this reads :

> " If one line meets another line, the sum of the adjacent angles so formed is equal to two right angles,"

and the proof given is a version of Euclid I, 13.

In the preface the author quotes Forder's *Euclidean Geometry* (1927) as showing that present-day standards of rigour are unthinkable in a school geometry, and adds that there must in fact be compromise. This is true enough. If he had consulted Forder's *School Geometry* (1930), which pays more attention to logic than any other school book in common use, he would have seen, for example, that his theorem 1 is there taken as assumption 1 (in accordance with the M.A. recommendations), and would have found on pp. 221, 222, substantial objections to his particular compromise.

No useful purpose would be served by enumerating the theorems which Mr. Roth proves, and for which by general agreement no proofs are now included in a Stage B course ; but it is hard to understand how an author who claims to give more than ordinary attention to logical principles can include the faulty and carelessly expressed proofs of the Theorems 35, 38, on tests for concyclic points.

One may well doubt whether in 1949 there remain any teachers who will want to teach this kind of formal geometry.

The historical notes, tactful discussion of logical niceties, systematic groupings, and glances forward and back, which are contained in the BY THE WAY sections, deal with matters which, rightly or wrongly, many teachers would omit. The topics are of the sort that the reviewer regards sympathetically, although he does not advocate Mr. Roth's restoration of incommensurable magnitudes into a school textbook. But there is so much in this book that must be attributed either to ignorance or carelessness that it is undesirable to say anything that might encourage a teacher to put the book into the hands of his pupils. Some of these deficiencies are only too clearly visible on the surface :

On p. 3 examples of the rules of logic are given which are, in fact, only rules of arithmetic or algebra. It is then suggested that a pleasant way to learn logic is to read *Alice in Wonderland* with care : a view apparently not shared by Lewis Carroll who thought it desirable to write also a separate book about logic.

A most unfortunate example of a " circular " definition is given and attributed to Euclid : " A point is that which has position but no size." This must have been taken from some modern substitute for Euclid, whose

words, Σημεῖον ἐστιν οὗ μέρος οὐθέν, will not bear that interpretation. We may regret that Euclid did not explicitly adopt Plato's view about points, but it is not easy to say, even with the help of a lexicon, just what was in Euclid's mind when he wrote the word ὅροι at the beginning of his work; and anyone who has even a Victorian schoolboy's acquaintance with the *Elements* will acquit the author of that work of circularity. The actual objection to Euclid's definition is that it only tells you something that a point is *not*.

An example of circularity can be found in one of Mr. Roth's niceties: "$8-7=1$, $13-12=1$, etc., are all true, but they are particular, whereas the algebraic statement $x-(x-1)=1$ not only contains them all, but verifies at one stroke every similar remark that we could possibly make". How is it proposed to verify $8-7=1$ from $x-(x-1)=1$ without using $8-1=7$?

Or again, "a *proof* consists of four sections, general enunciation, particular enunciation, construction, and *proof*" (our italics); but it is only fair to regard this as a piece of gross carelessness.

What meaning is the schoolboy expected to attach to Wallis' formula, stated in the form:

$$\frac{\pi}{2}=\frac{2^2 . 4^2 . 6^2 \ldots}{1^2 . 3^2 . 5^2 \ldots}?$$

Is this an intelligence test to see whether he will notice the corollary $\pi>8$, and on further reflection $\pi>14$?

The name *method of exhaustion* is applied in this book to a simple logical argument, whereas it is commonly used of the well-known process attributed to Eudoxus, which indeed includes the logic, but, much more than that, includes also the germ of the integral calculus.

We are told that a line divided so that $AX^2=BX . BA$ was said by the old geometers and philosophers to be in golden section. Small historical items are often interesting, but they should be accurate. What geometers or philosophers used that name before 1844?

Trigonometry is introduced on p. 322 after Theorem 59. Has Mr. Roth been influenced by Simson's *Euclid* (1781 edition), and does he seriously suggest either that this is the point at which trigonometry should be started, or that the three or four pages allotted to it indicate a satisfactory way of beginning the subject or serve any other useful purpose? He tells us that "the origin of the word sine is doubtful; probably like the word logarithm it is from the Arabic"; and also that "there is a simple rule for remembering the names of the three ratios. Thus:

"the sine is equal to *opposite side/hypotenuse*,"

etc., but how this helps one to remember the names is left unexplained. After finding the ratios for 60°, "using the same figure we can calculate the trigonometrical ratios of 30°; we have merely to turn the triangle round". What calculation is required, and why is the triangle turned round?

There is a list of seventeen books suitable for school libraries; and this expresses a curious preference for the *earlier* editions of Ball's *Recreations*. The author remarks, quite truly, that it is surprising how much a keen beginner can get out of books that should in theory be unintelligible to him: a good many beginners would be surprised at his seventeenth item, Russell's *Principles of Mathematics* (1903).

Readers of this notice will conclude that the reviewer thinks that this is a bad book. Such an opinion is necessarily individual, but it is undeniable that the author's treatment is contrary, in spirit and in practice, to the view of school geometry expressed in the two Reports of the Mathematical Association.

A. R.

An Introduction to the Study of Map Projection. By J. A. Steers. 7th edition. Pp. xxvii, 299. 18s. 1949. (University of London Press)

The publication of this new edition of a well-known book is an event of great importance. Professor Steers' work was written by a geographer for students of geography, and as such it should be judged : the mathematician, however, may also consider how far he finds it satisfying.

As a rule, those who proceed farthest in geography in schools have not studied mathematics beyond S.C. standard, and many teachers of the subject confess to little interest in map projections : indeed, a recent publication on the teaching of geography contains no reference to it, except a mere mention in a syllabus, and it is ignored in the Bibliography.

It therefore appears that the treatment of the subject should from necessity be practical rather than academic, though I incline to the view that it is the best beginning in any case, and the geography specialist should seek the help of his mathematical colleague.

This book is practical in approach, giving admirable directions and explanations. Teachers who find difficulty in affording the necessary time in the H.S.C. years might consider the possibility of spreading the work over the whole secondary school life without restriction to those who will ultimately specialise in geography. Its educational value is considerable, and it is surprising how much can be done, and with what enthusiasm, by quite young pupils using this book as a guide.

The Orthographic Map is described as of no great value : geographically, this is so, but, after all, it represents what is seen on a globe from a distance, and the obvious distortion round the periphery illustrates the problem involved in map projections. Furthermore, it is a fairly simple exercise on plan and elevation. I do not agree that the Calculus is necessary for the construction of Mercator's projection : the loxodrome can be drawn on any Orthomorphic projection—fairly easily on the Stereographic from the Pole—the longitudes of latitude intersections found and marked on the loxodrome drawn in advance across Mercator meridians. Among gnomonics mention might have been made of that produced by Messrs. Philip, starting with a solid cube and cutting off each corner by a section through mid-points of concurrent edges.

The book gives an account of all the projections commonly used in atlases, and is an excellent book of reference for less-known ones, as it has been brought up to date by the inclusion of many elegant examples produced for special purposes during recent years.

Professor Steers is modest about his mathematical attainments : but such a book could not have been produced without profound knowledge of the relevant mathematics, and its adaptation for the non-mathematical requires a high degree of skill.

A tribute is due to the excellent work of publisher and printer in producing a book worthy of the matter which it contains : the binding should ensure a long and useful life.

I recommend this book to all who take seriously the study of map projections. H.S.C. and University students should use it, and no geography or mathematics library will be complete without it. A. H.

Analytic Theory of Continued Fractions. By H. S. Wall. Pp. xiii, 433. 36s. 1948. University Series in Higher Mathematics. (Van Nostrand, New York ; Macmillan, London)

Continued fractions have not been very generously treated in English mathematical literature. The chapters which used to appear in every *Higher Algebra* have disappeared, while for a substantive treatise one must go to

the extraordinarily thorough *Kettenbrüchen* of Perron (2nd edition, 1929). The topic is nevertheless of intrinsic interest and of considerable importance in other parts of mathematical analysis; for instance, one of Ramanujan's most powerful weapons was his remarkable, perhaps unique, mastery of the manipulation of continued fractions.

In the new Van Nostrand series of texts on advanced mathematics, the first volume therefore deals with a subject on which a modern treatise in English is needed; and it sets a high standard for its promised successors. The book is clearly (though not quite flawlessly) printed, and is in many respects a model production. One feature of the text can be particularly commended as setting an example which could advantageously be followed by the further volumes of this series; that is, the provision in ten pages of a bird's-eye view of the contents. The reader who masters this will be considerably helped in tackling the detailed exposition which follows, for he will have a clear idea of the objectives which the author has in mind. In this detailed exposition, however, we may wish that the author had given us a little more variation of light and shade: the critical point of a chapter or the crux of an argument is not always clearly visible above the steady level of detail. If this comment is hypercritical, its defence is that the need for particular care on matters of "relief" is more than usually important in a book which is didactic rather than encyclopedic.

The book is divided into two parts, of which the first has the sub-title "Convergence Theory". The theory of the convergence of a continued fraction of complex elements was seriously investigated first by Pringsheim in memoirs of which the earliest is now about fifty years old. He established, for instance, the convergence of

$$\frac{a_1}{b_1+}\;\frac{a_2}{b_2+}\;\frac{a_3}{b_3+}\cdots$$

when the elements satisfy the condition

$$|\,b_r\,| \geqslant |\,a_r\,| + 1, \quad r = 1, 2, 3, \ldots .$$

At about the same time, a series of papers by Van Vleck, in the U.S.A., developed similar ideas; more recently, Professor Wall and his colleagues have been responsible for many extensions and generalisations, of which an account is given in the present volume. Although much has been accomplished, the reader may get the impression that finality has not yet been reached on this matter, and that some wide and unifying generalisation is yet to be found.

The second part, "Function Theory", the larger of the two sections, contains a number of applications of continued fractions to the theory of functions. But clearly the most important chapters here are those which owe their inspiration ultimately to Stieltjes, one of the most fruitfully creative of nineteenth-century mathematicians. Stieltjes was led by his studies on asymptotic series to develop a theory of continued fractions, to extend the notion of integration, and to study the famous "moment problem"; given a set of numbers c_n, to determine a function $f(t)$ such that

$$c_n = \int_0^\infty t^n\,df(t), \quad n = 0, 1, 2 \ldots .$$

Many years later, Hausdorff, in two powerful memoirs in *Mathematische Zeitschrift*, clarified much of the theory of summation processes by linking it with a moment problem in the interval (0, 1). Since then, Wall, Scott, Garabedian and others have studied the connection between continued fractions and summation processes, thus making the three lines of continued fractions,

moment problems, and summation processes concurrent, and this work is here set forth in a clear exposition.

Other topics are considered, but need not be mentioned here. It is sufficient to say that the new Van Nostrand series has been given a good start by a very clearly-written book on a fascinating topic, and that if this standard can be maintained in future volumes they will be sure of a warm welcome.

T. A. A. B.

The Theory and Use of the Complex Variable. 2nd edition. By S. L. Green. Pp. viii, 136. 12s. 6d. 1948. (Pitman)

This is a new edition of Mr. Green's very useful little textbook, wherein the student of science and technology will find a good guide to the use of the potent methods of the complex variable. Since the book is not intended for the professional mathematician, the space devoted to pure theory is restricted to the elements. Thus the existence of complex numbers is more or less taken for granted, Cauchy's theorem is proved by means of Green's formula under conditions which are sufficient for applications, the theoretical aspects of the calculus of residues, and its application to the calculation of definite integrals, are dismissed briefly. Space thus saved, and, from the point of view of the book, properly saved, is available for a detailed study of the geometrical representation of complex numbers, leading to a full investigation of the simpler conformal transformations, including that of Schwarz-Christoffel, so that the student can readily appreciate the importance of this theory in the study of two-dimensional potential problems. A short chapter, which is valuable but could perhaps with advantage have been longer, gives applications to the theory of alternating currents. An adequate but not excessive number of exercises for the student is provided.

Some of the theoretical details are a little unsatisfactory. For example, the argument which purports to prove that if $x+iy=0$, then $x=y=0$ (p. 5), seems to me fallacious, as in the nature of the case it is bound to be. But the value of the book lies in its emphasis on the " use " of the complex variable, and for this purpose it is excellent.

T. A. A. B.

Integraltafel. I. Unbestimmte Integrale. Von W. Gröbner und N. Hofreiter. Pp. viii, 166. 27s. 1949. (Springer, Vienna and Innsbruck)

It is a great pleasure to see once again the imprint of the great firm of Springer, which between the wars provided us with many mathematical classics. We hope that they will soon be contributing in full to the international effort needed to meet the urgent demands for mathematical text-books.

The present volume has been reproduced from manuscript, very clearly and indeed beautifully written,* listing something of the order of 2000 formulae. These are carefully arranged in three main sections. First we have 21 pages dealing with the rational integrand ; secondly, 85 pages of algebraic irrationals, where the irrational is the root of a linear function, the square root of a quadratic, or of the type giving rise to elliptic integrals ; thirdly, 60 pages of integrals involving the elementary transcendental functions in the integrand, including a final section on integrals of the Jacobian and Weierstrassian elliptic functions.

Such a list should be most valuable, provided it is reliable ; the checking of so many formulae would be a tremendous labour, and I have not attempted it, but random dips here and there suggest that errors cannot be frequent.

The second volume will contain definite integrals.

T. A. A. B.

* The manuscript forms for the hyperbolic sine and cosine seem at first sight to be likely to cause confusion with the ordinary " sin " and " cos ", but a very short time suffices to distinguish them.

Five-Figure Tables of Mathematical Functions. By J. B. DALE. 2nd edition. Pp. viii, 121. 6s. 1949. (Arnold)

Dale's tables provide in a handy and inexpensive form a very useful adjunct for the mathematician who has need of occasional reference to tables of the less common sort. The professional computer will usually want something more elaborate, and might consider 5 figures as an unsatisfactory compromise between 4 and 6 or 7. But the needs of those who may occasionally wish to know $P_4(0{\cdot}75)$, $\log \Gamma(1{\cdot}4)$, or the smallest zero of $J_0(x)$, are well served by this volume. In addition to the standard elementary tables of powers, logarithms, circular and hyperbolic functions, it has tables of complete and incomplete elliptic integrals (somewhat misleadingly termed elliptic functions), the gamma function, Bessel and Legendre functions, the exponential integrals, the probability integral, and various useful miscellaneous tables. The typography is good, old-style figures giving a clear reading even in a close setting, and the pages are reasonably free from unsightly " rules ". The eight pages of explanation might usefully have been more informative on the possibilities of interpolation. T. A. A. B.

Chambers's Six-Figure Mathematical Tables. By L. J. COMRIE. I. Logarithmic Values. Pp. xxii, 576. 42s. II. Natural Values. Pp. xxxvi, 576. 42s. 1949. (W. and R. Chambers)

In 1944 the familiar Chambers's seven-figure tables completed a century of service to mathematicians and computers of all kinds. In considering what revision was necessary in a new edition, the publishers asked advice from the outstanding authority of our own times, Dr. L. J. Comrie. On the basis of his report, and through the skill and industry of his Scientific Computing Service, a completely new " Chambers's " is before us, in two volumes, of which the contents are as follows :

VOLUME I. LOGARITHMIC VALUES.

Logarithms of numbers, 10,000(1)100,000 ; 8-figure logarithms, 1(1)1000; 8-figure logarithms, 1·00000(0·00001)1·10000 ; antilogarithms, 0(0·0001)1; Mx and x/M for $x=0(0{\cdot}001)1$.

Logarithms of trigonometrical functions of angles in degrees, minutes and seconds. Log sin and log tan, $0°(1'')1°\ 20'$; log sin, log cos, log tan and log cot, $0°(10'')10°(1')45°$.

Logarithms of trigonometrical functions of angles in degrees and decimals and in radians. Log cos of small angles and log sin of large angles ; log sin and log tan, $0°(0°{\cdot}001)5°$ and $0^r(0^r{\cdot}000\ 0175)0^r{\cdot}087$; log sin, log cos, log tan and log cot, $0°(0°{\cdot}01)45°$ and interval $0^r{\cdot}000\ 175$ throughout the quadrant; proportional parts for radians.

S and T functions for seconds of arc, minutes of arc, degrees and decimals, seconds of time, minutes of time, radians and hyperbolic functions ; logarithms of hyperbolic functions, $0(0{\cdot}001)3(0{\cdot}01)5(\text{various})\ \infty$; logarithms of the gamma function, $1(0{\cdot}001)2$; conversion tables ; proportional parts; bibliography of more extended tables ; physical constants ; mathematical constants.

VOLUME II. NATURAL VALUES.

Trigonometrical functions of angles in degrees, minutes and seconds. Auxiliary functions τ and σ, $0°–2°$; cotangents and cosecants, $0°(1'')0°\ 24'(10'')4°$; six trigonometrical functions, $0°(1')45°$.

Trigonometrical functions of angles in degrees and decimals. Auxiliary functions τ and σ, $0°–2°{\cdot}5$; cotangents and cosecants, $0°(0°{\cdot}001)1°$; six trigonometrical functions, $0°(0°{\cdot}01)45°$.

Circular functions, or trigonometrical functions with the argument in radians. Formulae; auxiliary functions τ and σ; reduction to the first quadrant; sines and cosines, $0^r(0^r{\cdot}1)50^r(1^r)100^r(100^r)5000^r$; six functions, $0^r(0^r{\cdot}001)1^r{\cdot}6$; cotangents and cosecants, $0^r(0^r{\cdot}0001)0^r{\cdot}1$.

Exponential and hyperbolic functions. Coth and cosech, 0(0·0001)0·1; formulae; auxiliary functions τh and σh; e^x, e^{-x}, sinh, cosh, tanh and coth, 0(0·001)3; e^{-x}, tanh and coth, 3(0·01)6(various) ∞; e^x, sinh and cosh, 3(0·001)6(various)100.

Natural logarithms, 1(0·001)10; Inverse circular and hyperbolic functions; the gudermannian and its inverse.

Powers, roots, reciprocals, factors and factorials, 1–1000; squares, cubes and factors, 1000–3400; prime numbers, 1–12919; conversion from rectangular to polar coordinates; gamma function, 1(0·001)2; the probability integral in different forms; interpolation formulae and coefficients; numerical differentiation; solution of differential equations; numerical integration; proportional parts for seconds; conversion tables; bibliography of more extended tables; physical constants; mathematical constants.

The change from the rather stumpy single volume to these two handsome tomes is merely a superficial indication of great internal changes. The first to be noticed is that the basic tables are now six-figure, not seven. For this change, there are two main reasons: one is that, in Dr. Comrie's opinion, backed up as it must be by most of us, four-figure and six-figure tables together suffice for the greater part of all computational requirements; the other, slightly more technical but at least as important, is that the main tables are effectively linear with six figures, a character which could not be maintained with seven figures save by an enormous increase in bulk. We next notice that the two volumes are devoted respectively to logarithmic and to natural values, and that this thus shows a considerable increase in the proportion of space given to natural values, an increase which corresponds roughly to the wider use of machines, for which the natural values supply the diet. Journalist hysteria may have suggested that all tables will soon be superseded by the fancifully-named modern machine calculators, but it is certain that this eventuality, happy or not, is not likely to occur for many years yet; I believe Dr. Comrie would say fifty years, and until then, at least, his tables are likely to remain the standard and final authority. It is perhaps superfluous to say that every care has been taken to make the tables fully adequate and easily readable; two related functions are both tabulated if it is thought that both are used sufficiently often, a small but typical instance being that in the tables connected with $\exp(-x^2)$ both forms of the integral are given, namely,

$$\frac{2}{\sqrt{\pi}}\int_0^x \exp(-t^2)\,dt \quad\text{and}\quad \sqrt{\left(\frac{2}{\pi}\right)}\int_0^x \exp(-\tfrac{1}{2}t^2)\,dt.$$

For dealing with cotangents and cosecants of small angles, auxiliary tables and auxiliary functions are both available. The latter method, of which Dr. Comrie has long been a strong advocate, gives $\tau = x \cot x$, for example, in the form of a "critical table", so that values for the slowly-varying function τ can be read off, and hence the value of $\cot x$ obtained by a division. Methods for using unfamiliar devices are explained in the introduction, and illustrated by examples. The introduction, the examples, the fund of miscellaneous information in the lists of constants, all add to the value and usefulness of the books.*

A casual glance conveys the impression of skilled typography; but a closer

* A delicate sense of humour even gives us the dates of that tiresomely *movable* east, Easter, on a page listing various *constants*.

study reveals the enormous amount of patience and craftsmanship which has been lavished on the printing. The spacing of lines and of digits has been carefully adjusted to produce the maximum clarity from the efficient old-style figures ; first differences, where given, are precisely interlined, and are italicised when a warning that second differences are not negligible is needed. In the main logarithm tables the two leading digits are given at every fifth line, but where these change, the rest of that line is printed in a Clarendon modern-style ; in tables where the change of a leading digit occurs as we pass down a column, consecutive entries give the old and new values, so that the correct figures are obtained by reading either up or down. Proportional parts are everywhere easily accessible. In spite of the amount of material carried, the pages are rarely heavy, the reason being that rules are avoided as much as possible so as to provide a clean, open page : a comparison with any heavily-ruled table will show how much the clean page is to be preferred.

The books lie flat at an opening, but it is possible that the binding may not be sufficiently robust ; this is, of course, merely the fault of post-war conditions of production.

So admirable a collaboration between computer, publisher and printer is a rare event. Dr. Comrie, I know, toils passionately for an ideal of perfection generally regarded as beyond the grasp of human fallibility ; Messrs. Chambers and Messrs. T. and A. Constable have evidently shared his enthusiasm, and between them they have made the ideal a reality. T. A. A. B.

Elementary Calculus and Allied Geometry. By J. Harvey. Pp. ii, 493. 25s. 1949. (Hutchinson's Scientific and Technical Publications)

As stated in the preface, the " range of the subject treated is that laid down in those parts of the syllabuses in Mathematics for B.Sc. (General) Science and Engineering degrees of London University which come under the title of the book ". Considerable space is given to geometry, which includes a brief discussion of conic sections and quadric surfaces, besides pedal curves, bipolar coordinates, evolutes and involutes and roulettes. There are some unusual figures of surfaces, for example, four on pp. 229–232, with equations such as

$$\{a(x-b)+z^2\}(kx+z)^2+e^4y^2=0.$$

The use of Fourier series and differential equations occupy the last two chapters. A harmonic analyser designed by the author is described in connection with Fourier series.

The book gives the impression that more care might have been exercised in planning it and in correcting the proofs. In the chapter on conics, capital letters are used for the coefficients in the general equation of the second degree, contrary to the accepted usage of small letters for the coefficients and capital letters for the cofactors of the elements in their determinant. A proof that continuity in an interval is uniform is sketched as early as p. 16, whereas only a trifling reference to implicit functions appears to be made (p. 62). In several places in the chapter on Fourier series, capital O is used for zero, in other places small o is used. F. B.

Fundamentals of Statistics. By J. B. Scarborough and R. W. Wagner. Pp. vii, 145. 13s. 1948. (Ginn and Co., Boston, U.S.A.)

This book was prepared to meet the need for a brief course in statistics for the post-war curriculum at the United States Naval Academy, since it was considered that no suitable textbook was already available. The projected course has not yet been given, but the book is offered to the public. A good deal of information is contained within its short compass, more or less of a didactic character, but it has the merit for students that in addition to

worked-out examples many problems are set for solution, the answers being provided at the end. The subject-matters dealt with in the various chapters are: Representation of Data, Averages, Measures of Dispersion, Comparison of Distributions, Correlation, Probability Functions, the Normal Curve and a Generalisation (*i.e.* the Gram-Charlier series) and Sampling; and there is an appendix on simple probability. The only table provided is one of the normal integral and its derivatives, for use in fitting the Gram-Charlier series.

The book has been constructed on the principle that mathematical results should be proved. Since, however, the readers are not assumed to have a very high standard of mathematical ability, a great deal is omitted which might be considered proper to the subject-matter of even an elementary course on statistics. It may be doubted whether a good course can be given nowadays which does not include something of what is generally called small sample distributions, but these have been left out for the sake of brevity. A notable omission is the use of the Sheppard correction to the calculated moments of a grouped frequency distribution. A good deal of space has been given to the fitting of frequency curves, particularly the Gram-Charlier, which is only one of the ways of dealing with data that depart from normality. It is the experience of the reviewer that the statistician does not often find himself concerned with this problem, but requires to know about theories of estimation and tests of significance. In any case, when he fits a frequency curve he requires to test the "goodness" of the fit, and has a handy tool here in Pearson's χ^2 test, which unfortunately is not mentioned.

So far as it goes this book should succeed in conveying to the reader something of what statistics tries to do, and part, at any rate, of the methods that have become standard. Its perusal should stimulate the reader to find out more from other sources.

J. WISHART.

The Third Mental Measurements Year-Book. Edited by O. K. BUROS. Pp. 1047. $12.50. 1949. (Rutgers University Press, New Brunswick)

This large book is a continuation of a series of publications relating to mental tests and reviews of tests which began in 1935, and continued in 1936, 1937, 1938, 1940 and 1941. Not all of these were Year-Books, and though the present volume deals with the period 1940–47, it is hoped that sufficient support will be given to enable a review of testing work to be published every two or three years.

The volume is divided into two main parts, (*a*) Tests and Reviews, (*b*) Books and Reviews. The former lists 663 tests, 713 original reviews by 320 reviewers, and 3,368 references on the construction, validity, use and limitations of specific tests. Here an attempt is made to list all commercially available tests—educational, psychological and vocational, published as separates in English-speaking countries between October 1940 and December 1947.

The aims as in the 1940 *Year-Book* are to make available up-to-date bibliographies of recent tests, to make readily available hundreds of critical test reviews, to assist test users in making better selections of the standard tests to meet their needs.

The second main section, "Books and Reviews", lists 549 books on measurements, and 785 excerpts from reviews of these books in 135 journals. The range of books is the same as that in the 1940 *Year-Book* with one exception. Books on the methodology of factor analysis are not included, but will be included in the next *Year-Book of Research and Statistical Methodology*.

Here the aims are to provide a full up-to-date list of books, with reviews of them, to enable test-users to make a selection more suited to their needs, and to stimulate in readers a more critical attitude, and indirectly to encourage a high standard of critical review-writing in this field of work.

The first section, on Tests and Reviews, to which three-quarters of the text is given, is subdivided into the following sub-sections : Achievement Batteries, Character and Personality, English, English Literature, English Vocabulary, Foreign Languages, French, etc., Intelligence—group, Intelligence—individual, Mathematics, Mathematics—Arithmetic, Mathematics—Calculus (1 item), Mathematics—Geometry (6 items), Miscellaneous.

Needless to say, the book is attractively produced, and easy to use. It should prove of great value to workers who require an easily accessible and informative survey of this rapidly enlarging field of research. F. W. W.

Elementary Statistical Analysis. By S. S. Wilks. Pp. xi, 284. (Lithoprinted.) 14s. 1948. (Princeton University Press ; Geoffrey Cumberlege, London)

S. S. Wilks was the author of *Mathematical Statistics* (reviewed in the *Mathematical Gazette*, December 1944, Vol. XXVIII, No. 282, p. 223) : the present volume refers to this earlier one as a " more advanced book on this subject ". The new book is prepared for a one-semester basic course (usually taken at Princeton in the freshman year). The thirteen chapter headings may give an indication of the scope of the work.

Chapters 1 to 3 are devoted to elementary descriptive statistics of a sample of measurements (1, Introductory ; 2, Frequency Distribution ; 3, Sample Mean and Standard Deviation). The algebra demanded of the reader is indicated by the fact that the standard formulae for permutations, combinations and the binomial coefficients are given on pp. 69–75. Euler diagrams are used in Chapter 3 for a demonstration of the occurrence of various combinations of events and of their complements : the chapter includes reference to variance and to degrees of freedom. Then come two chapters on probability and probability distributions : probability density functions are included in Chapter 5. Chapters 6 to 8 deal with the three distributions, the Binomial, the Poisson, and the Normal. In these, as in the earlier chapters, Wilks tends to attach more importance to the cumulative frequency curves than to the frequency curves themselves : in accordance with this practice, he gives as his table for the normal distribution the areas, and deals with the use of probability graph paper. The rest of the book relates to the application of sampling principles to the simpler problems of statistical inference. The chapter headings have been so far more or less on the traditional lines of other textbooks, although the treatment is original. But in Chapter 9 we have the beginnings of a new approach : it deals with Sampling, both from the mathematical and from the experimental directions. Chapter 10 breaks away from the traditional line entirely. It is entitled Confidence Limits of Population Sampling : it deals, *inter alia*, with Student's *t*, and the Clopper-Pearson (1934) *abac*. A short chapter on Statistical Significance Tests follows, with references to the idea of the Null Hypothesis. Then, in Chapter 12, Testing Randomness in Samples, we have sections on Runs (a subject, it will be remembered, Wilks has made his own) and Quality Control Charts. The last chapter (Ch. 13), Analysis of Pairs of Measurements, deals with regression, correlation coefficient (with the David *abac*) and multiple regression.

The book can be highly recommended. It is sound and scholarly. The questions are good, though there are no answers. The typescript itself seems to me to be much clearer than was that of *Mathematical Statistics* : perhaps this is because the earlier book included many heavy mathematical formulae. There are few misprints. There is a useful index but no bibliography. The book is based on some of Wilks' original work and is an original treatment of the subject : but though it is described as " elementary " it will put the reader on the right lines. Frank Sandon.

Algebra by Visual Aids. By G. P. MEREDITH. Edited by L. HOGBEN.

Book 1. The Polynomials. Pp. 160 + xxxiv, 43 charts. 10s.
Book 2. The Continuum. Pp. 122 + xvi, 39 charts. 8s. 6d.
Book 3. The Laws of Calculation. Pp. 132 + xxvi, 27 charts. 7s. 6d.
Book 4. Choice and Chance. Pp. 136 + xxii, 37 charts. 9s. 6d.
Answer Book. Pp. 96. 6s. 1948. (Allen and Unwin)

Those who are concerned lest visual aids shall " make it too easy for them " have little to fear from this book. For, whilst other visual experiments in mathematics have often sought, by a practical or environmental approach, for little more than appreciation, as a foundation for more serious work, here is a new approach to algebra which adds to the range of school algebra.

Book 1 deals with symbols, the formula, brackets, factors, the arithmetic series, simple equations, ratios, including the ratio used as a multiplier, and quadratic equations. It also introduces briefly directed numbers, permutations and the usual method of finding the arithmetic mean in statistics. Figurate number patterns, profusely illustrated by coloured charts, form the basis of much of this work.

Book 2 deals with graphs, leading from the histogram to the straight line, parabola and hyperbola and the use of graphs in solving equations. It also includes plotting by differences, the algebraic solution of the cubic, the harmonic series and the usual range of problems for algebraic solution.

Book 3 consists largely of manipulative work, multiplication, division, fractions, indices, surds, the geometric series and logarithms. There is an account of the history of counting and number systems, with exercises involving change from system to system and from one scale of notation to another.

Book 4 deals with permutations and combinations, the binomial expansion, the beginnings of finite differences, recurrent series, the biquadratic, the solution of equations by Horner's method and by iteration, the elements of probability (with special reference to playing cards) and the binomial distribution.

It is evident, from this list of topics, that the book proposes, not only a new approach to algebra, by way of the chart, the figurate number and series, but what is virtually a new syllabus for school algebra. But whereas most critics would wish to reduce the volume of manipulative work to be found in older books, it is here set up, in full strength, on a firmer foundation, with considerable addition of new work. The authors claim that the book contains all that is needed for School Certificate and much more ; although additional matter is starred, for omission according to taste, much of it is inherent in the sequence.

Those who will expect, after the pattern of *Mathematics for the Million*, an "Algebra for the Common People", will be disappointed. This is an Algebra for the scientist and the research worker. The stress is on statistics, on the solution of equations, on finite differences and on discrete quantities generally. The function, rapidly gaining ground through the new syllabuses which followed the *Jeffrey Report*, receives a setback.

A strong case is made out for these fundamental changes in the foreword, which is amply supported by the very readable text. It is common to regard elementary mathematical textbooks as little more than collections of exercises, for very few pupils read the text, except perhaps to search for a worked example when in difficulties. This book is written for the pupil to read. The explanations are profuse and in simple language, sometimes perhaps too much so. The first chapter, Fun with Figures, opens with ten pages on the use of symbols. Book 3 begins with nineteen pages on Numeral Systems. The numerous coloured charts are most attractive ; it is a pity they have

been segregated at the end of each book, involving frequent reference backwards and forwards. Most of the charts are extremely helpful, but occasionally they are little more than the normal blackboard work.

There are minor shocks for the teacher, as, for example, when he meets, in Book 2, the naive statement that $C \div \infty = 0$, or the introduction and rapid dismissal, in Book 3, of convergent and divergent series, to prepare the ground for the sum to infinity of a G.P. Not all will agree with the early introduction of the Σ notation (p. 36), the readiness to provide lists of rules of procedure, or the tendency in places to rely on intuition as a substitute for proof. Even so, every teacher will find much of value to him in the new approach, the new treatment of individual topics, and in the use of visual aids. It is a pity that the book is too expensive for most schools. It is to be hoped that this alone will not prohibit an experiment on these lines. I. R. V.

College Algebra. By Frederick S. Nowlan. Pp. xiv, 371. 18s. 1947. (McGraw-Hill)

This is a book of considerable interest for the student with a mathematical turn of mind. It is written for those who are starting a College course, and deals with Algebra from the beginning, but tackles it in a logical manner which would be only comprehensible to one of mature mind.

The first chapter is an introduction, with clear definitions of negative numbers and their manipulation. Professor Nowlan deals clearly with multiplication and with the commutative, distributive, and associative laws. There is a clearness and exactness which is refreshing; he is not trying to explain why processes work, as in a first reading, but states his definitions and draws his conclusions mathematically and logically. The same thoroughness is shown in his further development of notation in Ch. 8, where he introduces irrational numbers, and states the laws of exponents and definitions of fractional and negative indices. Again, in Ch. 19 he defines complex numbers by considering the complex $[a, b]$ and giving clear definitions of fundamental processes. He then shows, as with exponents, that previous definitions are included as special cases of the more extended work, with the real complex $[1, 0]$ corresponding to the real number 1.

The first nine chapters consist of what in England we term Elementary Mathematics up to the solution of quadratic equations. But it also includes the solution of simultaneous equations by determinants, introduced commendably early, and fractional and negative exponents. Thereafter the book follows normal lines, except that logarithms do not occur until Ch. 14, and then the interest is more in theoretical development than in numerical application. Progressions and the binomial theorem, the latter with fractional and negative indices, provide the only reference to series. The author misses a chance here when in finding the cube root of 30 he says that it is easier to use logarithms, and does not point out how much more accurately the work can be done by the use of the binomial theorem. Similarly, in the last chapter but one, on partial fractions, he fails to use the binomial theorem for expansions of functions which have been expressed in partial fractions.

There is a long and practical chapter on Interest and Annuities. Permutations and Combinations and Probability are treated briefly but clearly. In the theory of equations Prof. Nowlan treats symmetric functions of roots of equations of any degree, and then goes on to synthetic division and applies it to further solution of equations and approximate methods. He gives Descartes' rule of signs. The last chapter is a general treatment of determinants.

Examples occur throughout the chapters, and practical problems appear at the end of most chapters. The book, however, has a definitely theoretical rather

than practical bias, and for this reason its appeal is more to the mathematical scholar than to the practical engineer. The book seems unusual to an English reader, because it includes in a single volume such elementary and advanced work, the latter more normally occurring in a general course of Analysis. It is stimulating because of the clear and logical way it is presented. Schools could well have a copy in their mathematical library, since it gives a scholarly presentation of work pupils normally do, and does not go beyond topics which sixth-form pupils may tackle while at school.

The typography is excellent. Mistakes are almost non-existent. The reviewer discovered two small numerical mistakes. On p. 160, lines 6 and 7, the last figure but one should be 2, not 8. On p. 165, nine lines up, there is a 1 instead of $\bar{1}$.

K. S. SNELL.

Introduction to Dynamics. By MARTIN DAVIDSON. Pp. 128. 5s. 1949. (Winchester Publications)

This is the first of a series of school textbooks on physics, and is to be followed by companion volumes on statics and hydrostatics. It is written specially for those who are preparing for the London Matriculation and School Certificate examinations, and in eight short chapters it covers the normal elements of particle dynamics. However, pupils coming to the subject new are not likely to find the book easy reading, and the scarcity of examples is a real disadvantage ; there are only about five or six problems at the end of each chapter and a miscellaneous set of 27 at the end of the book.

After obtaining the equations of uniformly accelerated motion, the author discusses vertical motion under gravity ; he then introduces the parallelogram of velocities and applies it to relative velocity problems. Chapters on Newton's Laws of Motion follow, and there is a final chapter on work, energy and power.

Although the typography is clear, it is uniform throughout the book, which is unfortunate ; it would have helped the novice if, for instance, the statement of the problems worked out in the text had been printed in italics.

The discussion of the fundamental principles is well designed, but the examples are sometimes rather confusing, and occasionally there is some slackness in the argument and in the phraseology. For instance, (p. 35) a simple experiment is described " to show that all bodies fall with the same *velocity* . . ." (my italics), and then the author merely states : " As gravity acts continuously on a body allowed to fall towards the ground and not in an intermittent manner . . . the acceleration of a falling body *must* be uniform." (Newton's Laws are introduced later.)

Again, force and impulse are not clearly enough differentiated. We read (p. 71) : " Suppose we had two bodies, one twice the mass of the other, and that both are started off by two different forces which impart the same *velocity* to each body. . . . Hence force is measured not only by the mass of the body whose state it changes, but . . . it is also measured by the *change* in the velocity of the body." On the next page we read : " the *rate of change* of velocity is proportional to the impressed force."

Further, (p. 116) in dealing with a H.P. problem it is stated that : " If the maximum speed of the car is u ft./sec. *the force opposing motion* is $75u$ *ft. lb. per sec* " (my italics).

Also Fig. 33 is poor ; two tensions T_1 and two tensions T_2 should certainly be shown. Moreover, to solve the problem of moving weights connected by a string over a pulley by considering the total mass moved as acted upon by the difference of the weights is fundamentally unsound.

Among the few misprints and errors noted, one might mention the arithmetical slip at the top of p. 52, which invalidates the subsequent working.

J. TOPPING.

Problem Papers in Mechanics for Higher School Certificate Candidates. By A. S. Gosset Tanner and R. H. Cobb. Pp. 91. 5s. (A. S. Gosset Tanner, 115 Radbourne Street, Derby)

This is a collection of questions in Mechanics arranged "so that they advance by easy stages from School Certificate to Higher Certificate standard in six terms". There are four or five questions for each of the eight (!) weeks of each term, and the parts of the subject covered are listed at the beginning. The questions for the seventh, eighth and ninth terms are similarly arranged but are of scholarship or "advanced" standard. In addition there are sets of about twenty supplementary questions arranged term by term, and answers are included.

The collection therefore seems to cover adequately the requirements of Higher School Certificate candidates, and should prove useful to teachers and pupils alike.

A few small points might be mentioned; the plural "lbs." seems to have been retained in several places; on p. 23 an acceleration appears as 4 ft./(sec.) and question 16 (p. 73) might be reworded so that in the phrase "the attraction between two masses" the word "spheres" replaces "masses". Also it would be a help in using the book if the appropriate term was printed at the top of each page.

J. Topping.

The Lewis Carroll Puzzle Book; containing over 1,000 posers from *Alice in Wonderland* and other books by Lewis Carroll. Compiled by the Rev. D. B. Eperson. 32 pages, $6\frac{1}{2}'' \times 8\frac{1}{4}''$. Issued from the Appeal Office of the Bishop's Appeal Fund in the Diocese of Salisbury (to which the proceeds go), 97 Crane Street, Salisbury, Wiltshire, England. 2s. 6d.

This consists mainly of a number of "Quizzes" on various subjects—many of the questions, but not all, connected with Lewis Carroll (Rev. C. L. Dodgson) and his works. The mathematical part is short, but interesting: in it are some of Dodgson's "Pillow Problems" and simpler "Armchair Problems". The mathematics is not, however, quite confined to this section, but enters, though slightly, into the scoring of the marks for solutions in the game of "Syzygies" (see pages 17 and 28): it is interesting to study reasons for the scoring formula proposed!

The whole collection has an unusual freshness to those not well acquainted with Lewis Carroll's works—perhaps because both Dodgson and compiler are mathematicians.

J. C. P. Miller.

Applications physiques de la Transformation de Laplace. Par M. Parodi. Pp. viii, 177. 1948. Méthodes de calcul, B, 1. (Centre National de la Recherche Scientifique, Paris; Gauthier-Villars)

Under the general auspices of the Centre National de la Recherche Scientifique, the "Centre d'Études Mathématiques en vue des applications" is to publish treatises and monographs on applied mathematics, and on those aspects of mathematics which are of most importance to the mathematical physicist. M. Parodi's monograph is one of the first of such publications, and his topic is one with which every present-day mathematical physicist must have at least a bowing acquaintance. In adding to the already long list of books on the operational calculus, however, M. Parodi has found room for new results and for fresh emphasis on certain points of the theory.

The author begins with the fundamental formula

$$F(p)=\int_0^\infty e^{-pt} f(t)\,dt, \qquad \text{.....(a)}$$

but before developing any rules of manipulation he establishes the basic theory by showing, under fairly wide conditions,

(i) that for $\mathbf{R}(p) > 0$,

$$F(p) = \frac{1}{p}\left\{f(0+) + \frac{\mu(p)}{p}\right\},$$

where $\mu(p)$ is regular and bounded in the right-hand half-plane ;

(ii) that if $F(p)$ is of this form, the equation (a), as an equation for $f(t)$, has a solution, and that this solution is unique, given of course as the Bromwich integral.

The familiar rules of operation are then established formally, since proofs can present little difficulty if the preceding sections have been mastered, and a number of elementary transforms are derived, including those for " Morse dot " and " staircase " functions.

Chapters II and III apply the calculus to ordinary and partial differential equations, particularly those arising in circuit theory and in the conduction of heat.

Chapter IV indicates, briefly but very pleasantly, how the operational rules can be used to evaluate definite integrals and to establish many of the known formal properties of special functions, for instance, Bessel functions and Laguerre polynomials.

Chapters V and VI deal with matters which are obviously closely connected with the Laplace transform but have not received much attention in earlier books on the Laplace transform, namely integral equations, and Volterra's theory of composition. Volterra's law of composition for $f(x, y)$, $g(x, y)$,

$$f^{\star}g^{\star} = \int_x^y f(x, u)g(u, y)du$$

reduces, when f, g are functions of $y - x$ only, that is, functions of the closed cycle, to the product law of the Laplace calculus, that if $F(p)$ and $G(p)$ are the transforms of $f(t)$, $g(t)$, then $F(p)G(p)$ is the transform of

$$\int_0^t f(u)g(t-u)du.$$

For functions of the closed cycle, the two processes are equivalent.

The final chapter returns to physical applications in the study of electric networks ; the influence of J. R. Carson's firm and precise formulation of the theory in his noteworthy book, *Electric Circuit Theory*, is very marked.

The physicist of limited mathematical ability who nevertheless hankers after " Heaviside made easy " would find this book too difficult. But granted a fair grasp of the theory of contour integration, it makes a clear and readable account of a fascinating theory, with many applications to pure and applied mathematics. The new series has made an excellent beginning, and we anticipate with pleasure further volumes expounding the mathematical theories vital to modern physics. T. A. A. B.

Outline of the History of Mathematics. By R. C. Archibald. 6th edition. Pp. 114. \$1. 1949. Published as No. 2 of the Herbert Ellsworth Slaught Memorial Papers, *American Mathematical Monthly*, Vol. 56, No. 1, Part II.

The passing years have increased a little the girth of Professor Archibald's slim *Outline*, with 114 pages as against 62 pages of the third edition. Although the main text has been most thoroughly revised and worked over, the increase is principally in the Literature List and Notes, now 58 pages as against 12. Thus the booklet still remains the best short introduction to the history of mathematics up to the early years of the nineteenth century, while its value for reference purposes has been much increased. Teachers who wish to begin the study of the history of mathematics, either for its own sake or for the sake of classroom use, cannot do better than get this *Outline* ; from the main text

they can obtain a good general view of the main lines of mathematical development, from the Literature List and Notes they can obtain much useful detail and also authoritative guidance towards a more intensive study of any particular point which specially interests them. Our new Honorary Member must be warmly thanked for the work he has put into the preparation of this new edition. T. A. A. B.

Sport with Figures. By M. SAVAGE. Pp. 38. 3s. post free. (Mark Savage, Upper Basildon, Reading)

"Think of a number" problems, and similar tricks with digits, are old favourites with us all; their value as a stimulus to the arithmetical tyro is not to be despised. Mr. Savage has collected in his small pamphlet a number of items of this kind, age-finding, reversing of sums of money, curious properties of numerals. The "secret" of each trick is given, and the young arithmetician might be advised to study the "why" as well as the "how" of some of these. T. A. A. B.

Mathematics. Our great heritage. Essays selected and edited by W. L. SCHAAF. Pp. 291. $3.50. 1948. (Harper, New York)

The appearance, in increasing numbers, of books about mathematics suggests that a considerable reading public is seeking to know what mathematics is, what it does, what are its aims and what its relations to the general cultural and sociological development of mankind. Dr. Schaaf has had the happy idea of collecting and reprinting some essays which provide answers to these questions without too much technical apparatus. The essays are grouped into five sections: I. The creative spirit (J. W. N. Sullivan, G. H. Hardy, J. B. Shaw); II. Wellsprings (E. T. Bell, G. Sarton, D. J. Struik); III. The Queen (C. V. Newsom, C. G. Hempel, T. Dantzig); IV. The Handmaiden (T. Fort, J. W. Lasley, R. B. Lindsay, T. C. Fry); V. Humanistic bearings (Report of the Progressive Education Association, A. Henderson, A. Dresden, R. D. Carmichael).

The range is considerable; Sullivan and Hardy present mathematics as one of the great arts, Fry compels us to admire the role of mathematics in industry, Struik does his best to persuade us that mathematics is bourgeois and requires "conscious, planned reconstruction on the basis of materialist dialectics", Bell and Sarton trace the historic development of ideas.

It would be easy, and profitless, to enumerate essays which Dr. Schaaf might have included but did not. Seventeen others could perhaps be found to make as good but hardly a better volume. With this sincere commendation of a collection which should be useful to many readers—school libraries ought certainly to contain it—one small regret may perhaps be voiced, that some mathematicians expounding the cultural significance of mathematics have not realised that "fine writing" and pompous rhetoric merely weaken their argument. T. A. A. B.

PRINTED IN GREAT BRITAIN BY ROBERT MACLEHOSE AND CO. LTD.
THE UNIVERSITY PRESS, GLASGOW

levelop-
l detail
ny par-
Member
of this
A. A. B.

Savage,

are old
tyro is
number
ous pro-
young
"how"
A. A. B.

W. L.

ematics
mathe-
general
had the
answers
ays are
, G. H.
Struik);
Hand-
manistic
nderson,

atics as
natics in
ourgeois
aterialist

Schaaf
be found
nmenda-
libraries
ed, that
nematics
weaken
A. A. B.

Lib: Rob: Hooke.

GUILELMI OUGHTRED
AETONENSIS,
quondam Collegii Regalis
in CANTABRIGIA Socii,

CLAVIS MATHEMATICÆ
DENVO LIMATA,
Sive potius
FABRICATA.

Cum aliis quibusdam ejusdem
Commentationibus, quæ in sequenti pagina recensentur.

Editio tertia auctior & emendatior.

OXONIÆ,
Excudebat LEON. LICHFIELD, Vencunt
apud THO. ROBINSON. 1652.